UNIFORM BOUNDS FOR BLACK–SCHOLES IMPLIED VOLATILITY

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ABSTRACT. The Black–Scholes implied total variance function is defined by

$$V_{\rm BS}(k,c) = v \Leftrightarrow \Phi\big(-k/\sqrt{v} + \sqrt{v}/2\big) - e^k \Phi\big(-k/\sqrt{v} - \sqrt{v}/2\big) = c.$$

The new formula

$$V_{\rm BS}(k,c) = \inf_{x\in\mathbb{R}} \big[\Phi^{-1}\big(c + e^k\Phi(x)\big) - x\big]^2$$

is proven. Uniform bounds on the function $V_{\rm BS}$ are deduced and illustrated numerically. As a by-product of this analysis, it is proven that F is the distribution function of a log-concave probability measure if and only if $F(F^{-1}(\cdot)+b)$ is concave for all $b \geq 0$. From this, an interesting class of peacocks is constructed.

1. Introduction

Define the Black–Scholes call price function $C_{\rm BS} : \mathbb{R}\times[0,\infty)\to[0,1)$ by

$$\begin{aligned} C_{\rm BS}(k,v) &= \int_{-\infty}^{\infty} (e^{\sqrt{v}z - v/2} - e^k)^+ \phi(z)dz \\ &= \begin{cases} \Phi\big(-\frac{k}{\sqrt{v}} + \frac{\sqrt{v}}{2}\big) - e^k\Phi\big(-\frac{k}{\sqrt{v}} - \frac{\sqrt{v}}{2}\big) & \text{if } v>0 \\ (1-e^k)^+ & \text{if } v = 0, \end{cases}\end{aligned}$$

where $\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}$ is the standard normal density and $\Phi(x) = \int_{-\infty}^{x}\phi(z)dz$ is its distribution function. As is well known, the financial significance of the function $C_{\rm BS}$ is that the minimal replication cost of a European call option on a stock with strike K and maturity T in the Black–Scholes model [4] is given by the formula

$$\text{replication cost } = S_0 e^{-\delta T} C_{\rm BS}\left[\log\left(\frac{Ke^{-rT}}{S_0e^{-\delta T}}\right), \sigma^2 T\right]$$

where S_0 is the initial stock price, δ is the dividend rate, r is the interest rate and σ is the volatility of the stock. Therefore, in the definition of $C_{\rm BS}(k,v)$, the first argument k plays the role of log-moneyness and the second argument v is the total variance of the terminal log stock price.

Of the six parameters appearing in the Black–Scholes formula for the replication cost, five are readily observed in the market. Indeed, the strike K and maturity date T are specified by the option contract, and the initial stock price S_0 is quoted. The interest rate is the yield of a zero-coupon bond $B_{0,T}$ with maturity T and unit face value, and can be computed from

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the initial bond price via $B_{0,T} = e^{-rT}$. Similarly, the dividend rate can computed from the stock's initial time-T forward price $F_{0,T} = S_0 e^{(r-\delta)T}$.

As suggested by Latane & Rendleman [13] in 1976, the remaining parameter, the volatility σ, can also be inferred from the market, assuming that the call has a quoted price C^{quoted}. Indeed, note that for fixed k, the map $C_{\text{BS}}(k, \cdot)$ is strictly increasing and continuous, so we can define the inverse function

$$V_{\text{BS}}(k, \cdot) : [(1 - e^k)^+, 1) \to [0, \infty)$$

by

$$v = V_{\text{BS}}(k, c) \Leftrightarrow C_{\text{BS}}(k, v) = c.$$

The implied total variance of the call option is then defined to be

$$V^{\text{implied}} = V_{\text{BS}} \left[\log \left(\frac{K e^{-rT}}{S_0 e^{-\delta T}} \right), \frac{C^{\text{quoted}}}{S_0 e^{-\delta T}} \right],$$

and the implied volatility is

$$\sigma^{\text{implied}} = \sqrt{\frac{V^{\text{implied}}}{T}}.$$

Because of its financial significance, the function V_{BS} has been the subject of much interest. Unfortunately, there seems to be only one case where V_{BS} can be computed explicitly: when $k = 0$ we have

$$C_{\text{BS}}(0, v) = 2\Phi \left(\frac{\sqrt{v}}{2} \right) - 1$$

and hence

$$V_{\text{BS}}(0, c) = 4 \left[\Phi^{-1} \left(\frac{1+c}{2} \right) \right]^2.$$

In the general case, the function V_{BS} can be evaluated numerically as suggested by Manaster & Koehler [16] in 1982. Since then, there have been a number of approximations [5, 6, 15, 17] proposed as well as model-independent asymptotic formulae [2, 3, 7, 8, 9, 14, 19] for implied volatility.

The main purpose of this note is to give a new exact formula for V_{BS}:

Theorem 1.1. *For all $k \in \mathbb{R}$ and $(1 - e^k)^+ \le c < 1$ we have*

$$\begin{aligned} V_{\text{BS}}(k, c) &= \inf_{x \in \mathbb{R}} \left[\Phi^{-1}\big(c + e^k \Phi(x)\big) - x \right]^2 \\ &= \inf_{y \in \mathbb{R}} \left[y - \Phi^{-1}\big(e^{-k}(\Phi(y) - c)\big) \right]^2. \end{aligned}$$

Furthermore, if $c > (1 - e^k)^+$, then the two infima are attained at

$$x^* = -\frac{k}{\sqrt{v}} - \frac{\sqrt{v}}{2},$$

$$y^* = -\frac{k}{\sqrt{v}} + \frac{\sqrt{v}}{2}$$

where $v = V_{\text{BS}}(k, c)$.

Remark 1.2. We are using the convention that $\Phi^{-1}(u) = +\infty$ for $u \ge 1$ and $\Phi^{-1}(u) = -\infty$ for $u \le 0$.

The rest of the note is organised as follows. In section 2 the proof of Theorem 1.1 is presented. The main step is a duality Lemma 2.1 which may have some independent interest. In section 3 are some applications of Theorem 1.1 to give uniform bounds, strengthening existing asymptotic results. In section 4 these bounds are applied to study the shape of the implied volatility smile in the context of a general market model. In section 5, the bounds from section 3 are investigated numerically. Finally, in section 6, a generalisation of Lemma 2.1 gives rise to a characterisation of log-concave distributions. From this, an interesting class of peacocks (in the sense of Hirsh, Profeta, Roynette & Yor [10]) is constructed.

2. A duality formula for Black–Scholes call prices

The main result of this section is the following duality formula:

Lemma 2.1. *For all $k \in \mathbb{R}$ and $v \geq 0$ we have*

$$C_{\mathrm{BS}}(k, v) = \sup_{u \in [0,1]} \left[\Phi\big(\Phi^{-1}(u) + \sqrt{v}\big) - e^k u \right].$$

Proof. Fix $k \in \mathbb{R}$ and $v \geq 0$ and let

$$g(u) = \Phi\big(\Phi^{-1}(u) + \sqrt{v}\big) - e^k u.$$

If $v = 0$ then $g(u) = u(1 - e^k) \leq (1 - e^k)^+$ with equality if $u = \mathbf{1}_{\{k<0\}}$. So suppose $v > 0$. Note that the derivative is given by

$$\begin{aligned} g'(u) &= \frac{\phi\big(\Phi^{-1}(u) + \sqrt{v}\big)}{\phi\big(\Phi^{-1}(u)\big)} - e^k \\ &= \exp\big(-\sqrt{v}\Phi^{-1}(u) - v/2\big) - e^k. \end{aligned}$$

In particular $g'(u^*) = 0$ where

$$u^* = \Phi(-k/\sqrt{v} - \sqrt{v}/2).$$

Since g' is strictly decreasing, the function g is concave and hence

$$g(u) \leq g(u^*) = C_{\mathrm{BS}}(k, v) \quad \text{for all } u \in [0, 1].$$

□

Remark 2.2. The function $u \mapsto \Phi\big(\Phi^{-1}(u) + \sqrt{v}\big)$ appeared as the value function of the problem of maximising the probability of a perfect hedge considered by Kulldorff [12]. (Also see Section 2.6 of the book of Karatzas[11].) This function is increasing and concave, and as in the Lagrangian duality approach to utility maximisation, it is natural to compute its convex dual function. Lemma 2.1 says that this dual function is essentially the Black–Scholes call price function. We will return to this theme in section 6.

Proof of Theorem 1.1. Fix $k \in \mathbb{R}$ and $(1 - e^k)^+ \leq c < 1$ and let $v \geq 0$ be such that $C_{\mathrm{BS}}(k, v) = c$.

First, suppose $c > (1 - e^k)^+$ and hence $v > 0$. By Lemma 2.1 we have

$$\Phi\big(x + \sqrt{v}\big) - e^k \Phi(x) \leq c$$

with equality with

$$x^* = \Phi^{-1}(u^*) = -k/\sqrt{v} - \sqrt{v}/2.$$

Rearranging yields

$$\sqrt{V_{\mathrm{BS}}(k,c)} = \inf_{x\in\mathbb{R}} \left[\Phi^{-1}\big(c + e^k \Phi(x)\big) - x\right].$$

Since the left-hand side is non-negative, we have

$$\Phi^{-1}\big(c + e^k\Phi(x)\big) - x \geq 0$$

for all $x \in \mathbb{R}$. In particular, the square of the infimum is the infimum of the square, which yields the first expression. Let $x = \Phi^{-1}\left(e^{-k}(\Phi(y) - c)\right)$ in the above infimum to obtain the second expression.

The case where $c = (1-e^k)^+$ and hence $v = 0$ can be dealt with by continuity, or by noting that if $k \geq 0$ and $c = 0$, the map

$$x \mapsto \Phi^{-1}(e^k\Phi(x)) - x$$

is increasing and tends to 0 as $x \downarrow -\infty$. Similarly, if $k < 0$ and $c = 1 - e^k$ then the map

$$x \mapsto \Phi^{-1}(1 - e^k + e^k\Phi(x)) + x = -\Phi^{-1}(e^k\Phi(-x)) + x$$

is decreasing and tends to 0 as $x \uparrow +\infty$. $\square$

3. Uniform bounds

In their long survey article of [1], Anderson & Lipton warn that the many asymptotic implied volatility formulae that have appeared in recent years may not be applicable in practice, since typical market parameters are usually not in the range of validity of any of the proposed asymptotic regimes. The benefit of the new formula of Theorem 1.1 is that it yields upper bounds for implied volatility which hold *uniformly*, without any assumptions on parameter values. Indeed, the following bounds are completely model-independent since, of course, only the definition of the function V_{BS} is used.

In this section, in order to treat the cases $k \geq 0$ and $k < 0$ as symmetrically and efficiently as possible, we introduce some notation. First, let

$$P(k,c) = c + e^k - 1.$$

Note that if c is a call price, then $P(k,c)$ is the corresponding put price by put-call parity. Now we define a symmetrised call price by

$$\hat{C}(k,c) = \begin{cases} c & \text{if } k \geq 0 \\ e^{-k}P(k,c) & \text{if } k < 0. \end{cases}$$

This notation is a way to capture the Black–Scholes put-call symmetry identity

$$\hat{C}(k, C_{\mathrm{BS}}(k,v)) = C_{\mathrm{BS}}(|k|, v).$$

Many of the proposed asymptotic expansion in the literature rely on the fact that

$$\Phi(-x) = \frac{\phi(x)}{x}(1 + o(1))$$

as $x \uparrow +\infty$ and hence

$$\Phi^{-1}(\varepsilon) = -\sqrt{-2\log\varepsilon} + o(1)$$

or even more precisely

$$\left[\Phi^{-1}(\varepsilon)\right]^2 = -2\log\varepsilon - \log(-\log\varepsilon) - \log(4\pi) + o(1)$$

as $\varepsilon \downarrow 0$. Rather surprisingly, in several of the following examples, we will see that by replacing expressions of the form $-\sqrt{-2\log\varepsilon}$ in the asymptotic expansion with $\Phi^{-1}(\varepsilon)$ we essentially recover a uniform bound.

For example, Gulisashvili [9] showed that if $c(k) \downarrow 0$ as $k \uparrow +\infty$ then

$$\sqrt{V(k, c(k))} = \sqrt{2}\left(\sqrt{k - \log c(k)} - \sqrt{-\log c(k)}\right) + o(1).$$

Similarly, it was shown that if $e^{-k}p(k) \downarrow 0$ as $k \downarrow -\infty$ then

$$\sqrt{V(k, c(k))} = \sqrt{2}\left(\sqrt{-\log p(k)} - \sqrt{k - \log p(k)}\right) + o(1).$$

where $c(k) = 1 - e^k + p(k)$. A uniform version is this:

Corollary 3.1. *For all $k \in \mathbb{R}$ and $(1 - e^k)^+ \leq c < 1$, we have*

$$\begin{aligned}\sqrt{V(k, c)} &\leq \Phi^{-1}(2\hat{c}) - \Phi^{-1}(e^{-|k|}\hat{c}) \\ &= \begin{cases} \Phi^{-1}(2c) - \Phi^{-1}(e^{-k}c) & \text{if } k \geq 0 \\ \Phi^{-1}(2e^{-k}p) - \Phi^{-1}(p) & \text{if } k < 0 \end{cases}\end{aligned}$$

where $\hat{c} = \hat{C}(k, c)$ and $p = P(k, c)$.

Proof. When $k \geq 0$, let $x = \Phi^{-1}(e^{-k}c)$ in Theorem 1.1. And in the case $k < 0$, let $y = \Phi^{-1}(2 - e^k - c) = \Phi^{-1}(1 - p)$. □

For another example, in [19], it was shown that

$$V_{\mathrm{BS}}(k, c) = -8\log(1 - c) - 4\log[-\log(1 - c)] + 4k - 4\log\pi + o(1)$$

as $c \uparrow 1$. A uniform version of this asymptotic is this:

Corollary 3.2. *For all $k \in \mathbb{R}$ and $(1 - e^k)^+ \leq c < 1$, we have*

$$V_{\mathrm{BS}}(k, c) \leq 4\left[\Phi^{-1}\left(\frac{1 - c}{1 + e^k}\right)\right]^2$$

Proof. Let $x = \Phi^{-1}\left(\frac{1-c}{1+e^k}\right)$ in the Theorem 1.1. □

Remark 3.3. Note that since

$$\frac{1 - c}{1 + e^k} = \frac{1 - \hat{c}}{1 + e^{|k|}}$$

the conclusion of Corollary 3.2 can be rewritten as

$$V_{\mathrm{BS}}(k, c) \leq 4\left[\Phi^{-1}\left(\frac{1 - \hat{c}}{1 + e^{|k|}}\right)\right]^2,$$

to emphasise the symmetry between the $k \geq 0$ and $k < 0$ cases.

The next uniform bound of this section is inspired by a result of Lee [14]. The connection will be discussed in section 4.

Corollary 3.4. *For all $k \in \mathbb{R}$ and $(1-e^k)^+ \le c < 1$, we have*

$$\begin{aligned}\sqrt{V_{\mathrm{BS}}(k,c)} &\le \Phi^{-1}\left(\hat{c} + e^{|k|}\Phi(-\sqrt{2|k|})\right) + \sqrt{2|k|} \\ &= \begin{cases} \Phi^{-1}(c + e^k\Phi(-\sqrt{2k})) + \sqrt{2k} & \text{if } k \ge 0 \\ \Phi^{-1}(e^{-k}p + e^{-k}\Phi(-\sqrt{-2k})) + \sqrt{-2k} & \text{if } k < 0 \end{cases}\end{aligned}$$

where $\hat{c} = \hat{C}(k,c)$ and $p = P(k,c)$.

Proof. In the statement of Theorem 1.1, let $x = -\sqrt{2k}$ if $k \ge 0$, or let $y = \sqrt{-2k}$ if $k < 0$. □

We conclude this section with a lower bound that may be of interest, though it does not seem to arise directly from Theorem 1.1. Before we begin, we need to introduce the notation

$$\psi(k,x) = x + \sqrt{x^2 + 2|k|}$$

for $k, x \in \mathbb{R}$. We will also let $\psi(k,-\infty) = 0$ for all k. Note that $\psi(k,\cdot)$ is the inverse of the increasing function $y \mapsto y/2 - |k|/y$ on $[0,\infty)$.

The following lower bound can also be viewed as a uniform version of Gulisashvili's asymptotic formulae [9] and of the $c \uparrow 1$ formula appearing in [19].

Proposition 3.5. *For all $k \in \mathbb{R}$ and $(1-e^k)^+ \le c < 1$ we have*

$$\begin{aligned}\sqrt{V_{\mathrm{BS}}(k,c)} &\ge \psi\left[k, \Phi^{-1}(\hat{c})\right] \\ &= \begin{cases} \Phi^{-1}(c) + \sqrt{[\Phi^{-1}(c)]^2 + 2k} & \text{if } k \ge 0 \\ \Phi^{-1}(e^{-k}p) + \sqrt{[\Phi^{-1}(e^{-k}p)]^2 - 2k} & \text{if } k < 0, \end{cases}\end{aligned}$$

where $\hat{c} = \hat{C}(k,c)$ and $p = P(k,c)$.

Proof. Fix $k \in \mathbb{R}$ and $(1-e^k)^+ \le c < 1$ and $v = V_{\mathrm{BS}}(k,c)$. The result follows from the observation that

$$\begin{aligned}\Phi(-|k|/\sqrt{v} + \sqrt{v}/2) - \hat{c} &= e^{|k|}\Phi(-|k|/\sqrt{v} - \sqrt{v}/2) \\ &\ge 0.\end{aligned}$$

□

Remark 3.6. Note that we can quickly reprove Gulisashvili's formula when $c(k) \downarrow 0$ as $k \uparrow +\infty$ as follows: Using $\Phi^{-1}(\varepsilon) = -\sqrt{-2\log\varepsilon} + o(1)$ and Corollary 3.1 we have

$$\sqrt{V_{\mathrm{BS}}(k,c(k))} \le -\sqrt{-2\log c(k)} + \sqrt{2k - 2\log c(k)} + o(1)$$

and by Proposition 3.5 we have

$$\sqrt{V_{\mathrm{BS}}(k,c(k))} \ge -\sqrt{-2\log c(k)} + \sqrt{2k - 2\log c(k)} + o(1).$$

The $k \downarrow -\infty$ case is similar.

4. Applications to smile asymptotics

In this section, we apply the bounds in this previous section in the context of a general market model. We dispense with the full description of the market, and simply set

$$c(k) = \mathbb{E}[(X - e^k)^+]$$

where $X = S_T/F_{0,T}$ is a non-negative random variable with $\mathbb{E}[X] = 1$, modelling the ratio of the terminal stock price to its initial forward price under a fixed pricing measure.

In [14], Lee showed that

$$\limsup_{|k|\uparrow+\infty} \frac{V_{\mathrm{BS}}(k, c(k))}{|k|} \le 2.$$

The following proposition strengthens this.

Proposition 4.1.

$$\sqrt{V(k, c(k))} - \sqrt{2k} \to -\infty \ \textit{as}\ k \uparrow +\infty$$
$$\sqrt{V(k, c(k))} - \sqrt{-2k} \to \Phi^{-1}(\mathbb{P}(X = 0)) \ \textit{as}\ k \downarrow -\infty.$$

Proof. By Corollary 3.4 we have

$$\sqrt{V(k, c(k))} - \sqrt{2k} \le \Phi^{-1}(\hat{c}(k) + e^{|k|}\Phi(-\sqrt{2|k|})).$$

The standard bound on the normal Mills ratio yields

$$e^{|k|}\Phi(-\sqrt{2|k|}) \le \frac{1}{\sqrt{4\pi|k|}} \to 0 \text{ as } |k| \uparrow \infty.$$

Since $c(k) \downarrow 0$ as $k \uparrow \infty$ by the dominated convergence theorem, we have

$$\lim_{k\uparrow\infty} \sqrt{V(k, c(k))} - \sqrt{2k} \to -\infty$$

Also, we have for $k < 0$ that

$$\begin{aligned}\hat{c}(k) &= e^{-k}\mathbb{E}[(X - e^k)^+] - e^k + 1\\ &= \mathbb{E}[(1 - e^{-k}X)^+]\\ &\to \mathbb{P}(X = 0)\end{aligned}$$

as $k \downarrow -\infty$, and hence

$$\limsup_{k\downarrow-\infty} \left[\sqrt{V_{\mathrm{BS}}(k, c(k))} - \sqrt{-2k}\right] \le \Phi^{-1}(\mathbb{P}(X = 0)).$$

If $\mathbb{P}(X = 0) = 0$, then we are done. So suppose that $\mathbb{P}(X = 0) > 0$. Let $a(k) = \Phi^{-1}(e^{-k}p(k))$ and

$$a = \lim_k a(k) = \Phi^{-1}(\mathbb{P}(X = 0)) > -\infty.$$

Then by Proposition 3.5 we have

$$\begin{aligned}\sqrt{V(k, c(k))} - \sqrt{-2k} - a &\ge a(k) - a + \sqrt{[a(k)]^2 - 2k} - \sqrt{-2k}\\ &\to 0,\end{aligned}$$

completing the proof. ∎

Remark 4.2. The recent paper [7] of De Marco, Hillairet & Jacquier use the above proposition as a starting point for investigating the shape of the implied volatility smile in the case where $\mathbb{P}(X=0)>0$.

In [14], Lee also proved that if

$$p^* = \sup\{p \geq 0 : \mathbb{E}[X^{1+p}] < \infty\}$$

then

$$\limsup_{k\uparrow+\infty} \frac{V_{\mathrm{BS}}(k, c(k))}{k} = 2\left(\sqrt{p^*+1} - \sqrt{p^*}\right)^2,$$

and if

$$q^* = \sup\{q \geq 0 : \mathbb{E}[X^{-q}] < \infty\}$$

then

$$\limsup_{k\downarrow-\infty} \frac{V_{\mathrm{BS}}(k, c(k))}{-k} = 2\left(\sqrt{q^*+1} - \sqrt{q^*}\right)^2,$$

The following proposition is a uniform version of this.

Proposition 4.3. *Let $p \geq 0$ be such that $\mathbb{E}[X^{1+p}] < \infty$. Then for $k \geq 0$ we have*

$$\sqrt{V_{\mathrm{BS}}(k, c(k))} \leq \Phi^{-1}(2Me^{-pk}) - \Phi^{-1}(Me^{-(p+1)k})$$

where

$$M = \frac{p^p}{(1+p)^{1+p}}\mathbb{E}[X^{1+p}].$$

Similarly $q \geq 0$ be such that $\mathbb{E}[X^{-q}] < \infty$. Then for $k \leq 0$ we have

$$\sqrt{V_{\mathrm{BS}}(k, c(k))} \leq \Phi^{-1}(2Ne^{qk}) - \Phi^{-1}(Ne^{(q+1)k})$$

where

$$N = \frac{q^q}{(1+q)^{1+q}}\mathbb{E}[X^{-q}].$$

Proof. Lee [14] showed that

$$(X - e^k)^+ \leq (pe^{-k})^p \left(\tfrac{X}{1+p}\right)^{1+p}$$

and hence

$$c(k) \leq Me^{-pk}$$

for all k. The conclusion follows from the fact that $V_{\mathrm{BS}}(k,\cdot)$ is increasing and Corollary 3.1. Similarly, for the $k \leq 0$ case, Lee showed that

$$p(k) = 1 - e^k + c(k) \leq Ne^{(q+1)k}$$

and the result again follows from Corollary 3.1. □

5. Numerical results

In this section, we numerically test the bounds developed in section 3 in various regimes. By put-call parity, we need only consider the case $k \geq 0$.

$$R_1(k,c) = \Phi^{-1}(2c) - \Phi^{-1}(e^{-k}c)$$

$$R_2(k,c) = -2\ \Phi^{-1}\left(\frac{1-c}{1+e^k}\right)$$

$$R_3(k,c) = \Phi^{-1}\left(c + e^k\Phi(-\sqrt{2k})\right) + \sqrt{2k}$$

$$R_4(k,c) = \Phi^{-1}(c) + \sqrt{[\Phi^{-1}(c)]^2 + 2k}.$$

By the results of section 3 we have for all $k \geq 0$ and $0 \leq c < 1$ that

$$R_4(k,c) \leq \sqrt{V_{\mathrm{BS}}(k,c)} \leq R_i(k,c) \text{ for } i = 1, 2, 3.$$

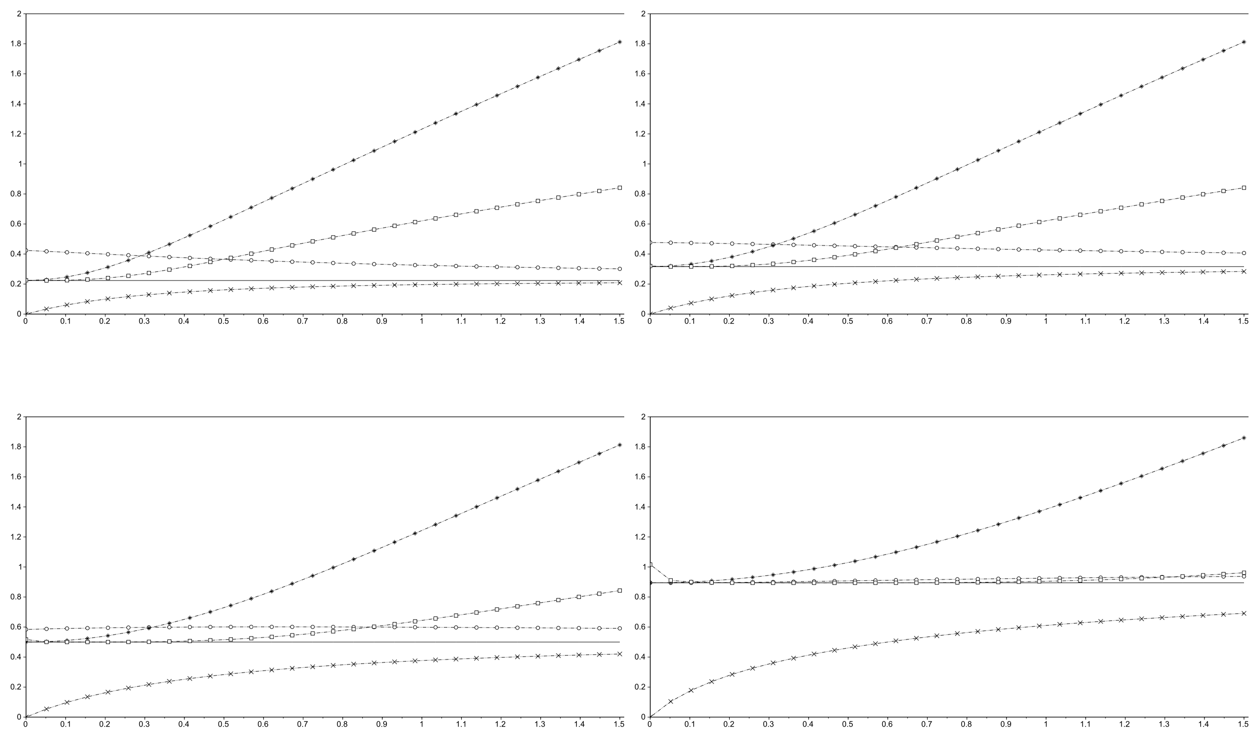

Figure 1. The graphs of k versus $R_i(k, c(k))$, where $\circ = R_1$, $* = R_2$, $\Box = R_3$ and $\times = R_4$, and $c(k) = C_{\mathrm{BS}}(k, v)$ for $v = 0.05$ (top left), $v = 0.10$ (top right), $v = 0.25$ (bottom left) and $v = 0.80$ (bottom right).

In Figure 5, the graphs of the functions $k \mapsto R_i(k, c(k))$ where

$$c(k) = C_{\mathrm{BS}}(k, v)$$

for various values of v. For comparison, the graphs of $k \mapsto \sqrt{V_{\mathrm{BS}}(k, c(k))} = \sqrt{v}$ are also plotted. It important to note that the graphs do not represent the implied volatility smile for a specific model, but rather they are a way to illustrate the various bounds over the parameter space (k, c).

It is interesting to observe that the upper bound R_2 is an extremely good approximation when the log-moneyness k is small. In fact, this is not surprising given the exact formula

$$\begin{aligned} V_{\text{BS}}(0, c) &= 4\left[\Phi^{-1}\left(\frac{1+c}{2}\right)\right]^2 \\ &= 4\left[\Phi^{-1}\left(\frac{1-c}{2}\right)\right]^2 \end{aligned}$$

mentioned in the introduction. The upper bound R_1 seems to be reasonably tight across a range of values of k, and of course, becomes better for large values of v. The upper bound R_3 is seen to be be most tight in the cases where k and v are about of the same size. Finally, the lower bound R_4 only seems to be a good approximation for large k. However, for fixed k, we have

$$\begin{aligned} R_4(k, c) &= 2\sqrt{-2\log(1-c)} + o(1) \\ &= R_2(k, c) + o(1) \end{aligned}$$

as $c \uparrow 1$, and hence this bound becomes tight for very large values of v.

6. Log-concave distributions and peacocks

The main observation of section 2 is that the Black–Scholes call pricing function C_{BS} is essentially (that is, in the variable $K = e^k$) the convex dual of the function $\Phi(\Phi^{-1}(\cdot) + \sqrt{v})$. Indeed, the main step in the proof of Lemma 2 is establishing the concavity of this function. In this section we explore a natural generalisation of this observation. It yields a curious characterisation of log-concave probality measures on the real line. We will also use this fact to construct a family of peacocks in the sense of Hirsh, Profeta, Roynette & Yor [10].

Theorem 6.1. *Let F be the distribution function of a probability measure with positive, continuous density f. The function*

$$u \mapsto F(F^{-1}(u) + b)$$

is concave for all $b \geq 0$ if and only if f is log-concave.

Proof. Fix $b \geq 0$ and let

$$G(u) = F(F^{-1}(u) + b)$$

Note that the derivative is given by the formula

$$G'(u) = \frac{f(F^{-1}(u) + b)}{f(F^{-1}(u))}.$$

Therefore, the function G is concave if and only if G' is decreasing, or equivalently,

$$\log f(y + b) - \log f(x + b) \leq \log f(y) - \log f(x)$$

for all $x \leq y$. This last condition holds for all $b \geq 0$ if and only if $\log f$ is concave. □

Let f be a positive log-concave density satisfying the Inada-like condition

$$\lim_{x\downarrow-\infty}\frac{f(x+b)}{f(x)}=+\infty \text{ and } \lim_{x\uparrow+\infty}\frac{f(x+b)}{f(x)}=0.$$

Inspired by Lemma 2.1, given a log-concave density f we can define a function $C_f:\mathbb{R}\times[0,\infty)\to[0,1)$ by

$$C_f(k,v)=\sup_{u\in[0,1]}F(F^{-1}(u)+\sqrt{v})-ue^k.$$

Of course we have $C_\phi=C_{\mathrm{BS}}$.

Notice that

- the map $v\mapsto C_f(k,v)$ is increasing with $C_f(k,0)=(1-e^k)^+$ for all $k\in\mathbb{R}$, and
- the map $K\mapsto C_f(\log K,v)$ is decreasing and convex for all $v\geq 0$.

Therefore, there exists a non-negative martingale $(M_v)_{v\geq0}$ defined on some filtered probability space such that

$$C_f(k,v)=\mathbb{E}[(M_v-e^k)^+].$$

See the paper of Roper [18] for details.

Now, define a function $U_f:\mathbb{R}\times[0,\infty)\to(0,1)$ by

$$U_f(k,v)=u\Leftrightarrow\frac{f(F^{-1}(u)+\sqrt{v})}{f(F^{-1}(u))}=e^k.$$

For instance, we have

$$U_\phi(k,v)=\Phi\left(\frac{-k}{\sqrt{v}}-\frac{\sqrt{v}}{2}\right).$$

Since it easy to see that the concave function $u\mapsto F(F^{-1}(u)+\sqrt{v})-ue^k$ is maximised at $u^*=U_f(k,v)$ we have the alternative formula

$$C_f(k,v)=F(F^{-1}(U_f(k,v))+b)-e^kU_f(k,v).$$

In particular, we can calculate the marginal distribution of M_v to get

$$\mathbb{P}(M_v\geq e^k)=U_f(k,v).$$

Recall that in the terminology of Hirsh, Profeta, Roynette & Yor [10], a peacock is a family $(\nu_t)_{t\geq0}$ of probability measures on $\mathbb{R}$ such that there is a one-dimensional martingale $(N_t)_{t\geq0}$ where the marginal law of N_t is ν_t for all $t\geq0$. Hence, the family of measures $(\mu_v)_{v\geq0}$ on $\mathbb{R}_+$ is a peacock, where μ_v is uniquely determined by

$$\mu_v[K,\infty)=U_f(\log K,v)$$

for $K>0$.

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