The Merton Problem with a Drawdown Constraint on Consumption

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Abstract

In this paper, we work in the framework of the Merton problem [17] but we impose a drawdown constraint on the consumption process. This means that consumption can never fall below a fixed proportion of the running maximum of past consumption. In terms of economic motivation, this constraint represents a type of habit formation where the investor is reluctant to let his standard of living fall too far from the maximum standard achieved to date. We state our candidate value function and optimal controls then provide a rigorous verification argument.

Keywords: Merton problem, Hamilton–Jacobi–Bellman equation, drawdown constraint, duality.

Mathematics Subject Classification: 49L20, 90C46, 91G10, 91G80.

JEL Classification: C61, G11.

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1 Introduction

The Merton problem – a question about optimal portfolio selection and consumption in continuous time – is indeed ubiquitous throughout the mathematical finance literature. Since Merton's seminal paper [17] in 1971, many variants of the original problem have been put forward and extensively studied to address various issues arising from economics. For example, Fleming and Hernández–Hernández [11] considered the case of optimal investment in the presence of stochastic volatility. Davis and Norman [6], Dumas and Luciano [8], and more recently Muhle-Karbe and co-authors [5], [13], [18] addressed optimal portfolio selection under transaction costs. Rogers and Stapleton [24] considered optimal investment under time-lagged trading. Vila and Zariphopoulou [26] studied optimal consumption and portfolio choice with borrowing constraints. The effects of different types of habit formation on optimal investment and consumption strategies have been explored in [3], [14], and [19]. Rogers [23] considers many interesting variations of the Merton problem.

A particular class of constrained optimal investment problems that forms an important and recurring theme in mathematical finance is optimal investment under a drawdown constraint. This constraint, roughly speaking, means that a certain process has to remain above a fixed proportion of the running maximum of its past values. Drawdown constraints on wealth have been studied by Elie and Touzi [10], and Roche [21]. Carraro, El Karoui, and Obłój [1], and Cherny and Obłój [2] studied drawdown constraints in more general semimartingale settings via Azéma–Yor processes. Grossman and Zhou [12] considered the problem of maximising the long-term growth rate of expected utility of final wealth, subject to a drawdown constraint.

The case we consider in this paper is the Merton problem with a drawdown constraint on consumption. Under this condition, the investor cannot let consumption fall below a fixed proportion of the running maximum of past consumption. In terms of economic motivation, this represents a type of habit formation where once the investor has reached a certain standard of living, he is reluctant to let his standard of living to fall too far from that level.

To be precise, we consider an agent who can invest in a risk-free bank account and a risky stock modelled by geometric Brownian motion. The agent seeks to maximise the expected infinite horizon discounted utility of consumption by finding the optimal portfolio selection and consumption strategies – subject to the drawdown constraint on consumption. As in the original Merton set-up we take the agent's utility function to be of constant relative risk aversion (CRRA).

We obtain our candidate value function and candidate optimal controls using a heuristic argument (included in the appendix) which involves solving the Hamilton-Jacobi-Bellman equation and transforming to dual variables. This is because the dual problem is significantly easier to handle and has an explicit analytic solution, which we invert to obtain our candidate value function and candidate optimal controls. To prove optimality, we modify the approach of Dybvig [9] (who considered the case where consumption is non-decreasing). We adapt methods used in Elie and Touzi [10] (who considered the wealth drawdown case) to prove that under the candidate optimal controls there exists a unique strong solution to the wealth equation. Our results show that the key parameter in this problem is the ratio of the investor's wealth to the running maximum of past consumption. For the optimal solution, we observe four different regions of behaviour based on the value of this parameter. For low values, consumption is restricted to the minimum level possible without violating the drawdown constraint. As the ratio increases, consumption increases with wealth. In the third region, we consume at the highest recorded level of consumption to date while we wait for the ratio to hit a critical level, after which we increase consumption to a new maximum. We specify the boundaries of these regions explicitly, as well as the optimal portfolio selection and consumption rules in each case.

This paper is organised as follows. In section 2, we outline the market model that we will be working in. In section 3, we state our candidate value function and candidate optimal controls then state the main results of this paper. In section 4, we give an intuitive explanation for the form of the optimal controls and provide plots of the value function and optimal controls. We also investigate numerically the effect of varying the drawdown constraint on the value function. Section 5 provides a rigorous verification argument to prove the optimality of our conjectured solution. In section 6, we give proofs for the technical lemmas stated in section 3 which concern the existence and uniqueness of a strong solution to the wealth equation under the optimal controls. In section 7, we give an argument to show that, just like in the standard Merton problem, the case we consider here is ill-posed for $R \leq R^*$ for a certain $0 < R^* < 1$ which we specify, where R represents the agent's coefficient of relative risk aversion. We give a brief conclusion in section 8. The heuristic argument for deriving our candidate value function and candidate optimal controls is included in the appendix.

2 Market model

We work in a filtered probability space, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, endowed with a standard Brownian motion, $W = (W_t)_{t\geq 0}$. For convenience, we define $\mathbb{E}_t[\cdot] = \mathbb{E}[\cdot|\mathcal{F}_t]$.

Our market set-up is the same as in the standard Merton problem. Formally, we have a risk-free bank account with constant interest rate, $r > 0$, and a risky stock, S, with price dynamics given by

$$dS_t = S_t(\sigma dW_t + \mu dt)$$

for constant volatility, $\sigma > 0$, and constant drift, $\mu \in \mathbb{R}$. To make the stock attractive to the investor, we assume that $\mu > r$.

We define an investment strategy to be an adapted process $\theta = (\theta_t)_{t\geq 0}$ taking values in $\mathbb{R}$ satisfying the integrability condition

$$\int_0^T \theta_t^2 dt < \infty \qquad \text{a.s. for all } T > 0. \tag{1}$$

A consumption strategy is an adapted process $c = (c_t)_{t\geq 0}$ taking values in $\mathbb{R}_+$ satisfying

$$\int_0^T c_t dt < \infty \qquad \text{a.s. for all } T > 0. \tag{2}$$

Given $y > 0$ and a consumption strategy c, define $Y^{y,c} = (Y^{y,c})_{t\geq 0}$ to be

$$Y_t^{y,c} = \max\left\{ Y_{0-}^{y,c}, \operatorname*{ess\,sup}_{0\leq s\leq t} c_s \right\} \tag{3}$$

where $Y_{0-}^{y,c} = y$. From a financial perspective, we can interpret $Y_{0-}^{y,c}$ in two ways: either as the maximum value of the investor's past consumption before time 0, or as the investor's preference for a particular standard of living in the future. Thus the process $Y^{y,c}$ represents the running maximum of the consumption process taking into account the investor's past consumption or future preference via the value of y.

Next, given $x > 0$ and a particular investment and consumption strategy (θ, c), our wealth process $X^{x,\theta,c} = (X_t^{x,\theta,c})_{t\geq 0}$ evolves according to the following wealth equation

$$dX_t^{x,\theta,c} = rX_t^{x,\theta,c} dt + \theta_t(\sigma dW_t + (\mu - r)dt) - c_t dt \tag{4}$$

for initial wealth $X_0^{x,\theta,c} = x$ and where

$$\begin{aligned} X_t^{x,\theta,c} &= \text{our wealth at time } t \\ c_t &= \text{our consumption at time } t \\ \theta_t &= \text{the wealth in the stock at time } t. \end{aligned}$$

At times we will drop the superscripts on $X_t^{x,\theta,c}$ and $Y_t^{y,c}$ when there is no ambiguity over which initial conditions and investment and consumption strategy we are referring to.

Finally, let $\mathcal{A}_b(x, y)$ denote the set of investment and consumption strategies (θ, c) which satisfy (1) and (2) together with the drawdown constraint on consumption

$$c_t \geq bY_t^{y,c} \quad \text{a.s.} \tag{5}$$

for all $t \geq 0$ and for a fixed $0 < b < 1$.

Remark 1. As we will see in Proposition 1, any $(\theta, c) \in \mathcal{A}_b(x, y)$ satisfies $X_t^{x,\theta,c} \geq 0$ almost surely for all $t \geq 0$ so we do not need to worry about including notions of admissibility in the definition of $\mathcal{A}_b(x, y)$.

Clearly, the case $b = 0$ is just the standard Merton problem, and taking $b = 1$ gives the special case where consumption is constrained to be non-decreasing. The $b = 1$ case was investigated by Dybvig [9] in 1995, and like the standard Merton problem it is possible to obtain an explicit solution in this case. However, taking $0 < b < 1$ gives a continuum of cases between these two extremes where the parameter b in a sense represents the willingness of the investor to sacrifice a proportion of his current standard of living in exchange for greater utility in the long-run.

In this paper, we will consider the following optimal investment and consumption problem

$$\sup_{(\theta,c)\in\mathcal{A}_b(x,y)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t) dt\right] \tag{†}$$

where $\rho > 0$ represents the agent's preference for the present and we take the agent's utility function, U, to be of constant relative risk aversion (CRRA), that is $U(x) = \frac{x^{1-R}}{1-R}$ for $R \neq 1$, and $U(x) = \log x$ for $R = 1$, where R is a positive real number which represents the investor's coefficient of relative risk aversion. For brevity, we exclude the case $R = 1$ from this paper because the analysis and results for this special case are very similar to the case $R \neq 1$.

Exactly as we see in the standard Merton problem, it turns out that for (†) to be finite we require $\gamma_M > 0$ where

$$\gamma_M = \frac{1}{R}\left[\rho - (1-R)\left(r + \frac{\kappa^2}{2R}\right)\right] \tag{6}$$

and $\kappa = \frac{\mu - r}{\sigma}$. This is equivalent to taking $R > R^*$ for a particular $0 < R^* < 1$ given by

$$R^* = \frac{1}{2r}\left[-\left(\rho - r + \frac{\kappa^2}{2}\right) + \sqrt{\left(\rho - r + \frac{\kappa^2}{2}\right)^2 + 2r\kappa^2}\right] \tag{7}$$

For more details on this, we refer the reader to section 7 where we demonstrate investment and consumption strategies that make (†) infinite for the case $R \leq R^*$.

3 Statement of results

In this section, we first make a few preliminary definitions before stating our main results which give the value function and optimal controls for (†) and also show existence and uniqueness of a strong solution to the wealth equation under these controls.

We obtained the value function and optimal controls via a heuristic argument that involved solving the Hamilton-Jacobi-Bellman equation and considering the dual formulation of the problem. We give this heuristic calculation in the appendix and only state the results in this section.

Define $J : \mathbb{R} \to \mathbb{R}$ as follows.

$$J(s) = \begin{cases} As^{1-R'} & \text{for} \quad 0 \leq s \leq a \\ Bs^{-\alpha} + Cs^{\beta} - \frac{1}{r}s + \frac{U(1)}{\rho} & \text{for} \quad a \leq s \leq 1 \\ Ds^{-\alpha} + Es^{\beta} + \frac{1}{\gamma_M}\tilde{U}(s) & \text{for} \quad 1 \leq s \leq b^{-R} \\ Fs^{-\alpha} - \frac{b}{r}s + \frac{U(b)}{\rho} & \text{for} \quad b^{-R} \leq s < \infty \end{cases} \tag{8}$$

where $R' = 1/R$ and

$$\tilde{U}(s) = \sup_{x>0}\{U(x) - xs\} = -\frac{s^{1-R'}}{1-R'}$$

$$C = \frac{\left(b^{1+R(\beta-1)} - 1\right)}{\beta(\alpha+\beta)} \left[\frac{1}{R\gamma_M}(R(\alpha+1)-1) - \frac{\alpha+1}{r}\right]$$

$-\alpha < 0 < 1 < \beta$ are the roots of

$$Q_1(t) = \frac{1}{2}\kappa^2 t(t-1) + (\rho - r)t - \rho$$

a is the root between 0 and 1 of

$$Q_2(t) = (\alpha+\beta)(R(\beta-1)+1)Ct^{\beta} - \frac{(\alpha+1)t}{r} + \frac{\alpha}{\rho}$$

$$A = \frac{a^{R'-1}}{\gamma_M}\left[\frac{1}{1-R} - a\right]$$

$$B = \frac{a^{\alpha}}{(\alpha+\beta)(R(\alpha+1)-1)}\left[\frac{\beta}{\rho} + \frac{(1-\beta)a}{r}\right]$$

$$D = B + \frac{1}{\alpha(\alpha+\beta)}\left[\frac{\beta-1}{r} - \frac{1}{R\gamma_M}(1+R(\beta-1))\right]$$

$$E = \frac{b^{1+R(\beta-1)}}{\beta(\alpha+\beta)}\left[\frac{1}{R\gamma_M}(R(\alpha+1)-1) - \frac{\alpha+1}{r}\right]$$

$$F = B + \frac{\left(1 - b^{1-R(\alpha+1)}\right)}{\alpha(\alpha+\beta)}\left[\frac{\beta-1}{r} - \frac{1}{R\gamma_M}(1+R(\beta-1))\right]$$

Define $v : \mathbb{R} \to \mathbb{R}$ as the dual transform of J:

$$v(z) = \inf_{0<s<\infty}\{J(s) + zs\} \tag{9}$$

It is straightforward though tedious to show that $J' < 0$ and $J'' > 0$. Thus the infimum on the righthand side of (9) is attained for $s = (J')^{-1}(-z)$. We refer the reader to Rockafellar [22] for more information on dual transforms.

Unfortunately, for $0 < b < 1$ it is not possible to invert J' analytically in all four regions so we cannot obtain v explicitly in all regions. However, we can obtain v explicitly for two of the four regions:

$$v(z) = \begin{cases} \frac{\left(z-\frac{b}{r}\right)^{1-R^*}}{1-R^*}(\alpha F)^{R^*} + \frac{U(b)}{\rho} & \text{for} \quad b/r \le z \le z_{b^{-R}} \\ U(z)\left[\frac{1}{-A(1-R')}\right]^{-1/R'} & \text{for} \quad z_a \le z < \infty \end{cases} \tag{10}$$

For the inner two regions, $z_{b^{-R}} \leq z \leq z_1$ and $z_1 \leq z \leq z_a$ we have to obtain v numerically. In the above, we use the notation $z_s = -J'(s)$.

Finally define our candidate value function $V : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ by

$$V(x,y) = y^{1-R} v\left(\frac{x}{y}\right) \tag{11}$$

for $x/y \geq b/r$. We now define our candidate optimal investment and consumption strategy. Let

$$\hat{\theta}(x,y) = -\frac{\mu - r}{\sigma^2}\frac{V_x(x,y)}{V_{xx}(x,y)} \quad \text{for } x/y \geq b/r \tag{12}$$

$$\hat{c}(x,y) = \begin{cases} by & \text{for} \quad b/r \leq x/y \leq z_{b^{-R}} \\ (V_x(x,y))^{-1/R} & \text{for} \quad z_{b^{-R}} \leq x/y \leq z_1 \\ y & \text{for} \quad z_1 \leq x/y \leq z_a \\ x/z_a & \text{for} \quad z_a \leq x/y < \infty. \end{cases} \tag{13}$$

Remark 2. As evidenced by the form of our candidate value function and candidate optimal investment and consumption strategy in (11), (12) and (13), the crucial parameter in this problem is x/y, the ratio of wealth to the running maximum of past consumption. In view of this, we define $Z_t^{x,y,\theta,c} = X_t^{x,\theta,c}/Y_t^{y,c}$ for $t \geq 0$ and we also define $z = x/y$ which explains our choice of notation in (9) and (10). An intuitive explanation of how this ratio affects the optimal controls is given in section 4.

Define $\mathcal{D}_{b/r} = \{(x,y) \in \mathbb{R}_+ \times \mathbb{R}_+ : x/y \geq b/r\}$. Note that the functions V, $\hat{\theta}$ and $\hat{c}$ are only defined for $(x,y) \in \mathcal{D}_{b/r}$. This is because, as we will see in Proposition 1, for any $(\theta, c) \in \mathcal{A}_b(x,y)$ we must have $(X_t^{x,\theta,c}, Y_t^{y,c}) \in \mathcal{D}_{b/r}$ almost surely for all $t \geq 0$.

Now fix $(x,y) \in \mathcal{D}_{b/r}$ and consider the following system of equations

$$dX_t = (rX_t - \hat{c}(X_t,Y_t))dt + \hat{\theta}(X_t,Y_t)(\sigma dW_t + (\mu - r)dt) \tag{14}$$

$$Y_t = \max\left\{Y_{0-}, \operatorname*{ess\,sup}_{0\leq s\leq t} \hat{c}(X_s,Y_s)\right\} \quad \text{for } t \geq 0 \tag{15}$$

where $(X_0, Y_{0-}) = (x,y)$. We are interested in proving the existence and uniqueness of a strong solution to (14) and (15). However, by the definition of $\hat{c}$ in (13), we have that $(X_t,Y_t)_{t\geq 0}$ is a solution to (14) and (15) if and only if $(X_t, Y_t)_{t\geq 0}$ is a solution to (14) and (16) as given below,

$$Y_t = \max\left\{y, \frac{\bar{X}_t}{z_a}\right\} \qquad \text{for } t \geq 0 \tag{16}$$

where $\bar{X}_t = \sup_{0\leq s\leq t} X_s$. Define $\mathcal{D}_a = \{(x,y) \in \mathbb{R}\times\mathbb{R}_+ : x/y \leq z_a\}$ and observe that by (16) we have that any solution $(X_t, Y_t)_{t\geq 0}$ to (14) and (16) is such that $(X_t, Y_t) \in \mathcal{D}_a$ almost surely for all $t \geq 0$. On the set $\mathcal{D}_a \backslash \mathcal{D}_{b/r}$, $\hat{\theta}$ and $\hat{c}$ are not defined but we extend them continuously here by setting $\hat{\theta}(x,y) = 0$ and $\hat{c}(x,y) = by$ so that $\hat{\theta}$ and $\hat{c}$ are defined on the whole of $\mathcal{D}_a$.

Remark 3. Extending $\hat{\theta}$ and $\hat{c}$ this way will not affect the form of the solution because as we will see in the proof of Lemma 3, the unique strong solution $(X_t, Y_t)_{t\geq 0}$ satisfies $(X_t, Y_t) \in \mathcal{D}_{b/r} \cap \mathcal{D}_a$ almost surely for $t \geq 0$.

We now present a few technical lemmas, the proofs of which are deferred to section 6.

Lemma 1. *$\hat{\theta}$ and $\hat{c}$ are Lipschitz on $\mathcal{D}_a$.*

Lemma 2. *The system of equations* (14) *and* (15) *has a unique strong solution for initial conditions $(X_0, Y_{0-}) = (x, y) \in \mathcal{D}_{b/r}$.*

For the next two results, we fix $(x, y) \in \mathcal{D}_{b/r}$ and set $(X_0, Y_{0-}) = (x, y)$. We let $(X_t, Y_t)_{t\geq 0}$ be the unique strong solution to (14) and (15) subject to these initial conditions. Finally, define $(\theta^*, c^*) = (\theta^*_t, c^*_t)_{t\geq 0}$ by $\theta^*_t = \hat{\theta}(X_t, Y_t)$ and $c^*_t = \hat{c}(X_t, Y_t)$ for $t \geq 0$.

Lemma 3. *$(\theta^*, c^*) \in \mathcal{A}_b(x, y)$.*

Our main result is given below. The proof is reported in section 5.

Theorem 1. *The investment and consumption strategy (θ^*, c^*) is an optimal solution to* (†)*. Moreover, the function V is the value function for* (†)*. That is*

$$V(x,y) = \sup_{(\theta,c)\in\mathcal{A}_b(x,y)} \mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t) dt\right]$$

for all $(x, y) \in \mathcal{D}_{b/r}$.

4 Analysis of results

In this section, we provide plots of the value function and the optimal investment and consumption strategies. We give an intuitive explanation for the form of the optimal controls and lastly, we investigate numerically the effect of varying b (i.e. tightening or loosening the drawdown constraint) on the value function.

In Figure 1, we plot the function, J, and its dual function, v, as well as the optimal controls, $\hat{\theta}$ and $\hat{c}$.

An intuitive explanation for the form of the optimal controls is as follows. Recall that we define $z = x/y$, the ratio of wealth to the running maximum of consumption. We also let $Z_t^{x,y,\theta^*,c^*} = X_t^{x,\theta^*,c^*}/Y_t^{y,c^*}$ and for convenience we drop the superscripts.

First consider $b/r \leq z \leq z_{b-R}$, which is the region where z is smallest. At $Z_t = b/r$, we have just enough wealth to maintain the drawdown constraint if we put all our wealth in the bank account and consume the interest, so $\theta_t^* = 0$ and $c_t^* = bY_t = rX_t$. As Z_t increases to z_{b-R} we still consume at the minimum allowed level, bY_t, but we have excess wealth which we invest in the risky stock.

As Z_t increases into the next region $z_{b-R} \leq z \leq z_1$, we now have more wealth compared to the running maximum of consumption so can afford to consume at a higher level. Thus c_t^* increases with Z_t until we have $c_t^* = Y_t$ which occurs at $Z_t = z_1$.

For $z_1 \leq z \leq z_a$, we do not yet have enough wealth to support consumption at a higher level so we keep our consumption constant at $c_t^* = Y_t$. However, we allow our investment in the risky stock θ_t^* to increase with Z_t.

In the final region, $z_a \leq z < \infty$, we have more than enough wealth to support consumption at the current maximum Y_t so the optimal action is to immediately increase Y_t until Z_t decreases to z_a. In actual terms this is achieved by having $Y_t = \max\{Y_{0-}, \bar{X}_t/z_a\}$ where $\bar{X}_t = \sup_{0\leq s\leq t} X_s$ so when Y_t is increasing, Z_t is actually held constant at $Z_t = z_a$.

In Figure 2, we provide a simulation of the stock price followed by plots of Z and the optimal controls, all against time, based on this simulation. The horizontal dashed lines in Figure 2b represent the critical values b/r, z_{b-R}, z_1, and z_a which give the boundaries of the four different regions of behaviour. As Z_t moves between these different regions, we can see the effect on the optimal consumption rule in Figure 2d. In the simulation, consumption initially varies with Z_t, then as Z_t increases, consumption is maintained at level Y_t. As Z_t increases further, Y_t is occasionally raised to keep $Z_t \leq z_a$. Finally as the stock price plummets, Z_t falls as well, so consumption drops until it hits bY_t and is maintained at that level so as not to violate the drawdown constraint.

Figure 3a shows the dual function, v, as a function of z for several values of b. Note that by (11), v is essentially a scaled version of the value function V. We clearly see that v decreases as b increases, because increasing b tightens the drawdown constraint, which in turn restricts the class strategies, $\mathcal{A}_b(x,y)$. Finally, Figure 3b plots $v(z)$ against b for several values of z. In this plot, we see once again how increasing b decreases the value of $v(z)$, as one expects.

5 Verification argument

In this section, we prove Theorem 1 by modifying the argument of Dybvig [9], which essentially uses the Davis–Varaiya Martingale Principle of Optimal Control [7]. We first prove a result that illustrates why it is sufficient to define V, $\hat{\theta}$ and $\hat{c}$ for $(x,y) \in \mathcal{D}_{b/r}$ as in (11), (12) and (13).

Proposition 1. *If $(\theta, c) \in \mathcal{A}_b(x,y)$ then $(X_t^{x,\theta,c}, Y_t^{y,c}) \in \mathcal{D}_{b/r}$ almost surely for all $t \geq 0$.*

Proof. Let $\zeta_t = \exp(-rt - \frac{1}{2}\kappa^2 t - \kappa W_t)$ denote the unique state-price density. Then it is a standard result (see Remark 9.3 page 137 in Karatzas and Shreve [16]) that $X_t^{x,\theta,c} \geq \mathbb{E}_t\left[\int_{s=t}^{\infty} \frac{\zeta_s c_s}{\zeta_t} ds\right]$ almost surely. Thus

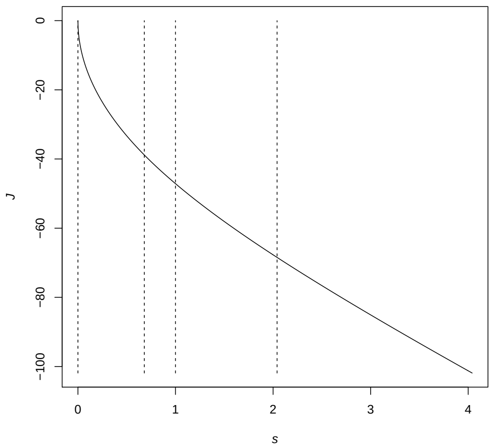
$$X_t^{x,\theta,c} \geq \mathbb{E}_t\left[\int_{s=t}^{\infty} \frac{\zeta_s b Y_t^{y,c}}{\zeta_t} ds\right] = \int_{s=t}^{\infty} e^{-r(s-t)} b Y_t^{y,c} ds = \frac{bY_t^{y,c}}{r} \quad \text{a.s.}$$

which gives the result. □

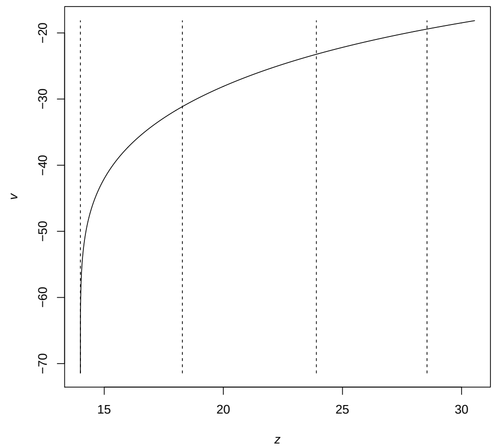
We will prove Theorem 1 via a series of lemmas. In what follows, we will always take $(X_0, Y_{0-}) = (x,y) \in \mathcal{D}_{b/r}$. Our first lemma is as follows.

Lemma 4. *The function V satisfies the Hamilton-Jacobi-Bellman (HJB) equation*

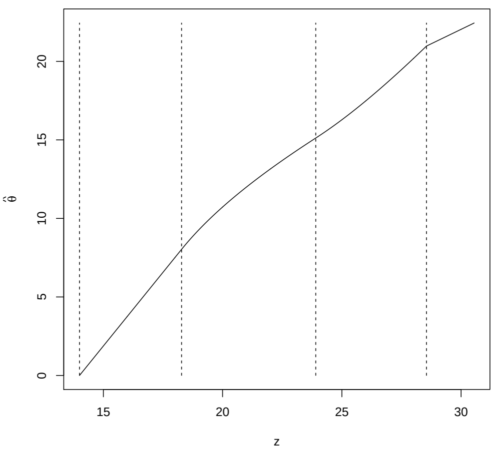
$$\max\left\{V_y, \sup_{\theta\in\mathbb{R}, c\geq by}\left[-\rho V + V_x(rx + \theta(\mu - r) - c) + \frac{1}{2}\sigma^2\theta^2 V_{xx} + U(c)\right]\right\} = 0. \tag{17}$$



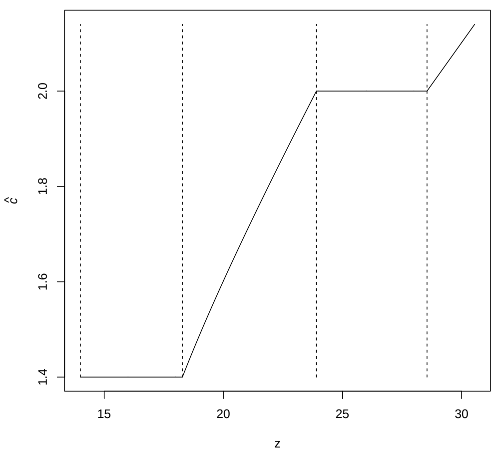
(a) J against s



(b) v against z

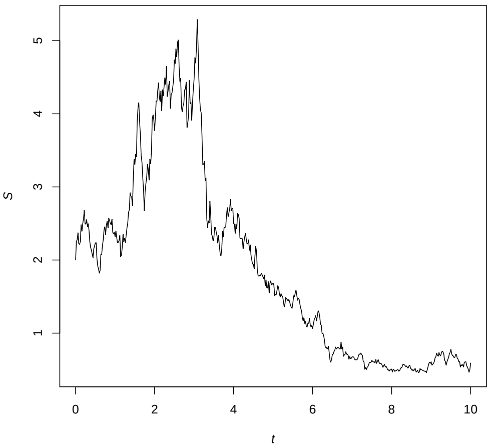


(c) $\hat{\theta}$ against z

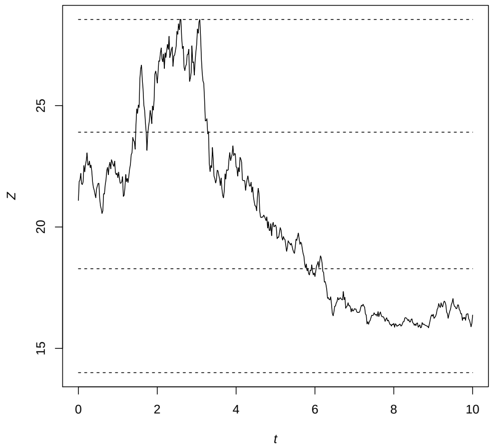


(d) $\hat{c}$ against z

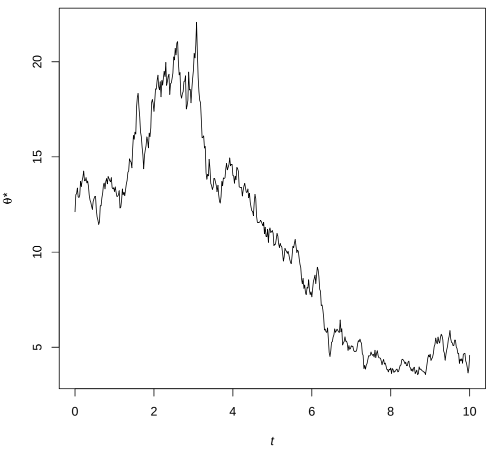
Figure 1: The vertical dashed lines represent the critical values b/r, z_{b-R}, z_1, and z_a which give the boundaries of the four different regions of behaviour. For all graphs we take $b = 0.7$, $y = 2$, $R = 2$, $\rho = 0.02$, $r = 0.05$, $\sigma = 0.35$, and $\mu = 0.14$.



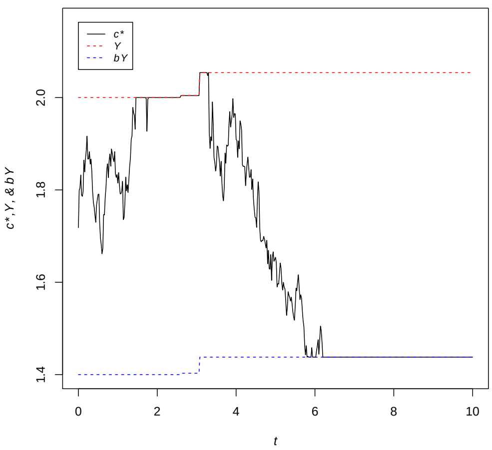
(a) Stock price, S, against t



(b) Z against t

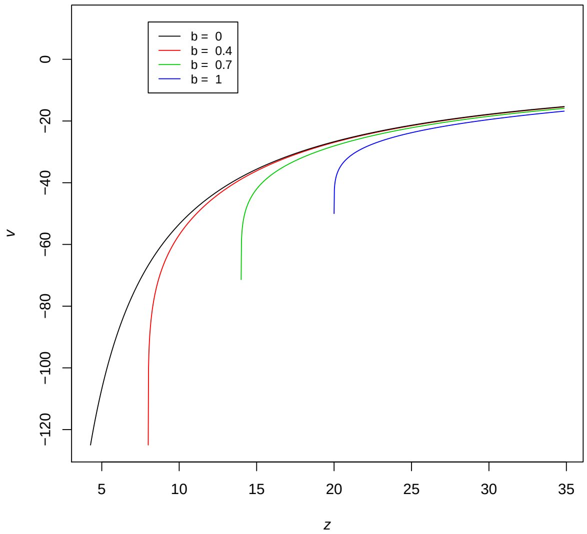


(c) θ^* against t

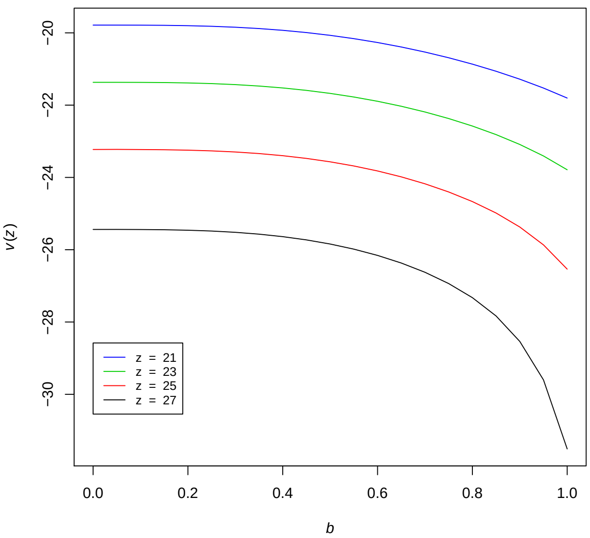


(d) c^*, Y, & bY against t

Figure 2: The above shows a simulation of the stock price and plots of Z and the optimal controls, θ^* and c^*, against time, t, based on this simulation. In Figure 2b, the horizontal dashed lines represent the critical values b/r, $z_{b^{-R}}$, z_1, and z_a which give the boundaries of the four different regions of behaviour. For all graphs we take $b = 0.7$, $y = 2$, $R = 2$, $\rho = 0.02$, $r = 0.05$, $\sigma = 0.35$, and $\mu = 0.14$.



(a) Dual function, v, against z for several values of b



(b) $v(z)$ against b for several values of z

Figure 3: In the above we take $y = 2$, $R = 2$, $\rho = 0.02$, $r = 0.05$, $\sigma = 0.35$, and $\mu = 0.14$.

Proof. By direct verification using (8), one can check that J satisfies

$$\begin{array}{rcll} 0 & = & (1-R)J + RsJ' & \text{for} \quad 0 < s \le a \\ 0 & = & -\rho J + (\rho - r)sJ' + \frac{1}{2}\kappa^2 s^2 J'' + U(1) - s & \text{for} \quad a \le s \le 1 \\ 0 & = & -\rho J + (\rho - r)sJ' + \frac{1}{2}\kappa^2 s^2 J'' + \tilde{U}(s) & \text{for} \quad 1 \le s \le b^{-R} \\ 0 & = & -\rho J + (\rho - r)sJ' + \frac{1}{2}\kappa^2 s^2 J'' + U(b) - bs & \text{for} \quad b^{-R} \le s < \infty. \end{array} \tag{18}$$

Note that since J is twice continuously differentiable, so is its dual v. Using the link between the derivatives of J and its dual v (we refer the reader to Rockafellar [22] for more details) this implies that

$$\begin{array}{rcll} 0 & = & -\rho v + rzv' - \frac{1}{2}\kappa^2 \frac{(v')^2}{v''} + U(b) - bv' & \text{for} \quad b/r \le z \le z_{b^{-R}} \\ 0 & = & -\rho v + rzv' - \frac{1}{2}\kappa^2 \frac{(v')^2}{v''} + \tilde{U}(v') & \text{for} \quad z_{b^{-R}} \le z \le z_1 \\ 0 & = & -\rho v + rzv' - \frac{1}{2}\kappa^2 \frac{(v')^2}{v''} + U(1) - v' & \text{for} \quad z_1 \le z \le z_a \\ 0 & = & (1-R)v - zv' & \text{for} \quad z_a \le z < \infty. \end{array} \tag{19}$$

Finally by (11), we have

$$\begin{array}{rcll} 0 & = & -\rho V + rxV_x - \frac{1}{2}\kappa^2 \frac{V_x^2}{V_{xx}} + U(by) - byV_x & \text{for} \quad b/r \le x/y \le z_{b^{-R}} \\ 0 & = & -\rho V + rxV_x - \frac{1}{2}\kappa^2 \frac{V_x^2}{V_{xx}} + \tilde{U}(V_x) & \text{for} \quad z_{b^{-R}} \le x/y \le z_1 \\ 0 & = & -\rho V + rxV_x - \frac{1}{2}\kappa^2 \frac{V_x^2}{V_{xx}} + U(y) - yV_x & \text{for} \quad z_1 \le x/y \le z_a \\ 0 & = & V_y & \text{for} \quad z_a \le x/y < \infty. \end{array} \tag{20}$$

From (8), it is straightforward although surprisingly tedious to verify that $(1-R)J + RJ's \le 0$, $J' < 0$ and $J'' > 0$. By (9) and (11), this is equivalent to $V_y \le 0$, $V_x > 0$ and $V_{xx} < 0$. This together with (20) implies that V satisfies the HJB equation (17). □

To proceed further, we need the following definition.

Definition 1. For $(\theta, c) \in \mathcal{A}_b(x, y)$, define

$$\xi_t^{x,y,\theta,c} = \int_0^t e^{-\rho s} U(c_s) ds + e^{-\rho t} V(X_t^{x,\theta,c}, Y_t^{y,c}). \tag{21}$$

We have the following result.

Lemma 5. *For any $(\theta, c) \in \mathcal{A}_b(x, y)$, $(\xi_t^{x,y,\theta,c})_{t \ge 0}$ is a local supermartingale. For (θ^*, c^*), $(\xi_t^{x,y,\theta^*,c^*})_{t \ge 0}$ is a local martingale.*

Proof. By Itô's formula,

$$d\xi_t^{x,y,\theta,c} = \sigma\theta_t V_x dW_t + \mathcal{L}^{\theta,c} V\,dt + V_y dY_t^{y,c} \tag{22}$$

where

$$\mathcal{L}^{\theta,c} V(x,y) = -\rho V + V_x(rx + \theta(\mu - r) - c) + \frac{1}{2}\sigma^2\theta^2 V_{xx} + U(c). \tag{23}$$

For any $(\theta, c) \in \mathcal{A}_b(x,y)$, Lemma 4 and (22) imply that $\xi^{x,y,\theta,c}$ is equal to a local martingale plus a decreasing drift term, thus $\xi^{x,y,\theta,c}$ is a local supermartingale.

Now consider (θ^*, c^*). By (16), Lemma 3 and Proposition 1, we have that $(X_t^{x,\theta^*,c^*}, Y_t^{y,c^*}) \in \mathcal{D}_{b/r} \cap \mathcal{D}_a$ almost surely for all $t \geq 0$ or equivalently that $b/r \leq Z_t^{x,y,\theta^*,c^*} \leq z_a$ almost surely for all $t \geq 0$. For $b/r \leq Z_t^{x,y,\theta^*,c^*} < z_a$, Y_t^{y,c^*} is constant so $V_y dY_t^{y,c^*} = 0$ and we also have $\mathcal{L}^{\theta^*,c^*} V = 0$ by (20). Note that by (16), when Y_t^{y,c^*} is increasing we have that Z_t^{x,y,θ^*,c^*} is held constant at z_a at which point $V_y = 0$ and $\mathcal{L}^{\theta^*,c^*} V = 0$ by (20). Putting this all together gives that ξ^{x,y,θ^*,c^*} is equal to a local martingale plus a zero drift term, which gives the result. □

The next step is to strengthen the conclusion of the above lemma from local (super)martingale to (super)martingale. To do this, we first need to prove a result about the wealth process.

Lemma 6. *Fix $p \neq 0$. We have that*

$$\mathbb{E}\left[\left(X_t^{x,\theta^*,c^*}\right)^p\right] \leq x^p \exp(\tilde{b}t) \tag{24}$$

for a constant $\tilde{b}$ that depends on p and the parameters of the problem.

Proof. For convenience, we drop the superscripts on X_t^{x,θ^*,c^*} and Y_t^{y,c^*}. An application of Itô's formula to $\log X_t^p$ gives

$$\begin{aligned} X_t^p \;=\; & x^p \exp\left(\int_{s=0}^t p\left(r + \frac{\theta_s^*}{X_s}(\mu - r) - \frac{c_s^*}{X_s} + \frac{1}{2}(p-1)\left(\frac{\theta_s^*}{X_s}\right)^2 \sigma^2\right) ds\right) \\ & \times \exp\left(\int_{s=0}^t p\frac{\theta_s^*}{X_s}\sigma dW_s - \frac{1}{2}\int_{s=0}^t p^2 \left(\frac{\theta_s^*}{X_s}\right)^2 \sigma^2 ds\right). \end{aligned} \tag{25}$$

We wish to show that we can choose a constant $\tilde{b}$ to bound the integrand in the first exponential. However, as we showed in the proof of Lemma 5, under

(θ^*, c^*), we have that $(X_t, Y_t) \in \mathcal{D}_{b/r} \cap \mathcal{D}_a$ almost surely for all $t \geq 0$. Thus, it is sufficient to show that the integrand in the first exponential is bounded for $(x, y) \in \mathcal{D}_{b/r} \cap \mathcal{D}_a$. To do this, it is enough to show that $\hat{\theta}(x, y)/x$ and $\hat{c}(x, y)/x$ are bounded for $(x, y) \in \mathcal{D}_{b/r} \cap \mathcal{D}_a$. Note that $\hat{\theta}(x, y)/x = \hat{\theta}(x, y)/y \times y/x$ and $\hat{c}(x, y)/x = \hat{c}(x, y)/y \times y/x$. Clearly, on $\mathcal{D}_{b/r} \cap \mathcal{D}_a$, $\hat{c}(x, y)/y$ is bounded between 0 and 1, and $y/x \in [1/z_a, r/b]$. Finally, observe that by (9), (11) and (12)

$$\frac{\hat{\theta}(x, y)}{y} = -\frac{\mu - r}{\sigma^2} \frac{v'(x/y)}{v''(x/y)} = \frac{\mu - r}{\sigma^2} s J''. \tag{26}$$

By (10), we can check directly that (26) is bounded for $b/r \leq x/y \leq z_{b^{-R}}$. By (8), we can check directly that (26) is bounded for $a \leq s \leq b^{-R}$ which is equivalent to $z_{b^{-R}} \leq x/y \leq z_a$. This gives that $\hat{\theta}(x, y)/y$ is bounded for $(x, y) \in \mathcal{D}_{b/r} \cap \mathcal{D}_a$ as desired. Putting this all together gives that we can choose a constant $\tilde{b}$ which depends on p and the parameters of the problem that bounds the integrand in the first exponential.

Hence $\mathbb{E}[X_t^p] \leq x^p \exp(\tilde{b}t)$ since the second term in (25) is a non-negative local martingale thus is a supermartingale so has expected value less or equal to 1. □

With the above result in hand, we can strengthen the conclusion of Lemma 5 to:

Lemma 7. *For any $(\theta, c) \in \mathcal{A}_b(x, y)$, $(\xi_t^{x,y,\theta,c})_{t \geq 0}$ is a supermartingale. For (θ^*, c^*), $(\xi_t^{x,y,\theta^*,c^*})_{t \geq 0}$ is a martingale.*

Proof. To show the first part of the lemma, given that by Lemma 5 $(\xi_t^{x,y,\theta,c})_{t \geq 0}$ is a local supermartingale, it is enough to show that $\xi^{x,y,\theta,c}$ is bounded below. This is because is it easy to check that a local supermartingale bounded below is a supermartingale. The fact that $V_x > 0$ (see proof of Lemma 5) together with Proposition 1 implies that

$$V(X_t^{x,\theta,c}, Y_t^{y,c}) \geq V\left(\frac{bY_t^{y,c}}{r}, Y_t^{y,c}\right) = U\left(bY_t^{y,c}\right)/\rho \geq U(by)/\rho > -\infty. \tag{27}$$

Hence, V is bounded below. Using this, we can show that $\xi^{x,y,\theta,c}$ is bounded below as follows.

$$\xi_t \geq \int_0^t e^{-\rho s} U(by) ds + e^{-\rho t} U(by)/\rho = \frac{U(by)}{\rho} > -\infty \quad \text{a.s.}$$

which gives that $\xi^{x,y,\theta,c}$ is a supermartingale for any $(\theta, c) \in \mathcal{A}_b(x, y)$.

Now for the second part of the lemma. Given Lemma 5, we just need to show that $\mathbb{E}\langle \xi^{x,y,\theta^*,c^*}\rangle_t < \infty$ for all $t \geq 0$, since this would imply that the local martingale ξ^{x,y,θ^*,c^*} is in fact a true martingale (see Corollary 1.25 in [25]). In the remainder of the proof, we drop the superscripts on ξ_t^{x,y,θ^*,c^*}, X_t^{x,θ^*,c^*}, Y_t^{y,c^*} and Z_t^{x,y,θ^*,c^*} for convenience. Under (θ^*, c^*)

$$d\xi_t = -\kappa e^{-\rho t}\frac{V_x^2}{V_{xx}}dW_t = \kappa e^{-\rho t}Y_t^{1-R}s_t^2 J'' dW_t$$

where we have used (9) and (11) and where $s_t = (J')^{-1}(-Z_t)$. From the definition of J in (8), it is straightforward to check that $s^2 J''$ is bounded on $[a, \infty)$ say $|s^2 J''| \leq M$ for some constant $M > 0$. Thus

$$\mathbb{E}\langle \xi \rangle_t \leq M^2\kappa^2 \int_0^t \mathbb{E}\left(Y_s^{2(1-R)}\right) ds \tag{28}$$

where the use of Fubini's theorem is justified because the integrand is positive. Now recall that we take $R > R^*$ as mentioned in Section 2. Thus we have the following two cases.

- $R^* < R < 1$: We have $Y_s \leq \frac{rX_s}{b}$ almost surely from Proposition 1. This implies that

  $$\mathbb{E}\left(Y_s^{2(1-R)}\right) \leq \left(\frac{r}{b}\right)^{2(1-R)} \mathbb{E}\left(X_s^{2(1-R)}\right) \leq \left(\frac{r}{b}\right)^{2(1-R)} x^{2(1-R)} \exp(\tilde{b}s)$$

  using the bound given by Lemma 6 taking $p = 2(1-R)$. Substituting this into (28) gives that $\mathbb{E}\langle \xi \rangle_t < \infty$ for all $t \geq 0$.

- $R > 1$: We have that Y is an increasing process and $Y_0 \geq y > 0$ by (3). Thus $Y_s^{2(1-R)} \leq y^{2(1-R)}$ and substituting this into (28) once again gives $\mathbb{E}\langle \xi \rangle_t < \infty$ for all $t \geq 0$.

In both cases, $\mathbb{E}\langle \xi^{x,y,\theta^*,c^*}\rangle_t < \infty$ for all $t \geq 0$ which implies that ξ^{x,y,θ^*,c^*} is a martingale. □

As a final step, we now address the asymptotic behaviour of the residual term $\mathbb{E}[e^{-\rho t}V(X_t^{x,\theta,c}, Y_t^{y,c})]$ for $(\theta, c) \in \mathcal{A}_b(x, y)$. To do this we adapt the argument given in Lemma 6 in Dybvig [9].

Lemma 8. *For any* $(\theta, c) \in \mathcal{A}_b(x, y)$

$$\liminf_{t\to\infty} \mathbb{E}[e^{-\rho t}V(X_t^{x,\theta,c}, Y_t^{y,c})] \geq 0. \tag{29}$$

For (θ^*, c^*)

$$\lim_{t\to\infty} \mathbb{E}[e^{-\rho t}V(X_t^{x,\theta^*,c^*}, Y_t^{y,c^*})] = 0. \tag{30}$$

Proof. For any $(\theta, c) \in \mathcal{A}_b(x, y)$, by (27) we have that

$$\liminf_{t\to\infty} \mathbb{E}[e^{-\rho t} V(X_t^{x,\theta,c}, Y_t^{y,c})] \geq \lim_{t\to\infty} e^{-\rho t} U(by)/\rho = 0.$$

Now, for (θ^*, c^*), we will consider the cases $R > 1$ and $R^* < R < 1$ separately. For $R > 1$, we have $J(0) = 0$ hence by (9) we deduce that $v(z) \leq 0$ for all $z \geq b/r$ which in turn implies that $V \leq 0$ by (11). But we just showed (29) so we must have (30) as required. Finally for $R^* < R < 1$, recall that $(X_t^{x,\theta^*,c^*}, Y_t^{y,c^*}) \in \mathcal{D}_{b/r} \cap \mathcal{D}_a$ almost surely for $t \geq 0$ by (16), Lemma 3 and Proposition 1, and that $V_y \leq 0$ by (17). This together with (10) and (11) implies that for $(x, y) \in \mathcal{D}_{b/r} \cap \mathcal{D}_a$ we have

$$V(x, y) \geq V\left(x, \frac{rx}{b}\right) = \frac{r^{1-R} x^{1-R}}{\rho(1-R)}.$$

Once more using that $V_y \leq 0$ by (17) we see that for $(x, y) \in \mathcal{D}_{b/r} \cap \mathcal{D}_a$ we have

$$V(x, y) \leq V\left(x, \frac{x}{z_a}\right) = \frac{x^{1-R}}{1-R}\left[\frac{1}{-A(1-R')}\right]^{-1/R'}$$

where the equality is by (10) and (11). Hence, to obtain (30) it is enough to show that

$$\mathbb{E}\left[e^{-\rho t}(X_t^{x,\theta^*,c^*})^{1-R}\right] \to 0 \text{ as } t \to \infty.$$

Taking $p = 1 - R$ in (25) and dropping the superscript on X_t^{x,θ^*,c^*} gives

$$\begin{aligned}
&\mathbb{E}\left[e^{-\rho t} X_t^{1-R}\right] \\
&= x^{1-R}\mathbb{E}\left[\exp\left(\int_0^t (1-R)\left(r + \frac{\theta_s^*}{X_s}(\mu - r) - \frac{c_s^*}{X_s} - \frac{R}{2}\left(\frac{\theta_s^*}{X_s}\right)^2 \sigma^2\right) - \rho\, ds\right)\right. \\
&\qquad \left.\times\ \exp\left(\int_0^t (1-R)\frac{\theta_s^*}{X_s}\sigma dW_s - \frac{1}{2}\int_0^t (1-R)^2\left(\frac{\theta_s^*}{X_s}\right)^2 \sigma^2 ds\right)\right] \\
&\leq x^{1-R}\mathbb{E}\left[\exp\left(\int_0^t \left((1-R)\left(r + \frac{\kappa^2}{2R}\right) - \rho\right) ds\right)\right. \\
&\qquad \left.\times\ \exp\left(\int_0^t (1-R)\frac{\theta_s^*}{X_s}\sigma dW_s - \frac{1}{2}\int_0^t (1-R)^2\left(\frac{\theta_s^*}{X_s}\right)^2 \sigma^2 ds\right)\right]
\end{aligned}$$

where the quadratic form $(\mu - r)\frac{\theta_s^*}{X_s} - \frac{R\sigma^2}{2}\left(\frac{\theta_s^*}{X_s}\right)^2$ in $\frac{\theta_s^*}{X_s}$ was replaced by its maximum value $(\mu - r)^2/2\sigma^2 R$ and $\frac{c_s^*}{X_s}$ was replaced by 0, a lower bound. Also, note that the second exponential term is a non-negative local martingale so is a supermartingale thus has expected value less or equal to 1. Hence

$$\begin{aligned}
\mathbb{E}\left[e^{-\rho t}(X_t^{x,\theta^*,c^*})^{1-R}\right] &\leq x^{1-R}\exp\left(-\left(\rho-(1-R)\left(r+\frac{\kappa^2}{2R}\right)\right)t\right) \\
&= x^{1-R}\exp(-R\gamma_M t) \\
&\to 0 \qquad \text{as } t\to\infty
\end{aligned}$$

since γ_M (defined in (6)) is strictly positive by assumption (see section 2). $\square$

We are now ready to provide a proof of our main result, Theorem 1.

Proof of Theorem 1. We need to show that for (θ^*, c^*)

$$V(x,y) = \mathbb{E}\left[\int_{t=0}^{\infty} e^{-\rho t}U(c_t^*)dt\right]$$

and also that for any $(\theta, c) \in \mathcal{A}_b(x,y)$,

$$V(x,y) \geq \mathbb{E}\left[\int_{t=0}^{\infty} e^{-\rho t}U(c_t)dt\right].$$

By Lemma 7, we have that under (θ^*, c^*), ξ^{x,y,θ^*,c^*} is a martingale which gives

$$\begin{aligned}
V(x,y) &= \xi_0^{x,y,\theta^*,c^*} \\
&= \lim_{t\to\infty}\mathbb{E}[\xi_t^{x,y,\theta^*,c^*}] \\
&= \lim_{t\to\infty}\mathbb{E}\left[\int_{s=0}^{t} e^{-\rho s}U(c_s^*)ds + e^{-\rho t}V(X_t^{x,\theta^*,c^*}, Y_t^{y,c^*})\right] \\
&= \lim_{t\to\infty}\mathbb{E}\left[\int_{s=0}^{t} e^{-\rho s}U(c_s^*)ds\right] + \lim_{t\to\infty}\mathbb{E}[e^{-\rho t}V(X_t^{x,\theta^*,c^*}, Y_t^{y,c^*})] \\
&= \mathbb{E}\left[\int_{t=0}^{\infty} e^{-\rho t}U(c_t^*)dt\right]
\end{aligned}$$

where we have used Lemma 8, and exchanging the order of the expectation and the limit is justified by $U(c_s^*) \geq U(by) > -\infty$.

To complete the proof observe that by Lemma 7, for any $(\theta, c) \in \mathcal{A}_b(x,y)$,

$\xi^{x,y,\theta,c}$ is a supermartingale, hence

$$
\begin{aligned}
V(x,y) &= \xi_0^{x,y,\theta,c} \\
&\geq \lim_{t\to\infty} \mathbb{E}[\xi_t^{x,y,\theta,c}] \\
&= \lim_{t\to\infty} \mathbb{E}\left[\int_{s=0}^{t} e^{-\rho s}U(c_s)ds + e^{-\rho t}V(X_t^{x,\theta,c}, Y_t^{y,c})\right] \\
&= \lim_{t\to\infty} \mathbb{E}\left[\int_{s=0}^{t} e^{-\rho s}U(c_s)ds\right] + \lim_{t\to\infty} \mathbb{E}[e^{-\rho t}V(X_t^{x,\theta,c}, Y_t^{y,c})] \\
&\geq \mathbb{E}\left[\int_{t=0}^{\infty} e^{-\rho t}U(c_t)dt\right]
\end{aligned}
$$

where we have used Lemma 8, and exchanging the order of the expectation and the limit is justified by Fatou's lemma. This completes the proof. □

6 Proof of technical lemmas

In this section we provide the proofs of Lemmas 1, 2 and 3, which mostly concern the existence and uniqueness of a strong solution to the wealth equation under the optimal controls.

Proof of Lemma 1. Recall that for $(x,y) \in \mathcal{D}_a \backslash \mathcal{D}_{b/r}$ we set $\hat{\theta}(x,y) = 0$ and $\hat{c}(x,y) = by$ to extend $\hat{\theta}$ and $\hat{c}$ continuously to the whole of $\mathcal{D}_a$. The partial derivatives of $\hat{\theta}$ and $\hat{c}$ with respect to x and y are clearly bounded on $\mathcal{D}_a \backslash \mathcal{D}_{b/r}$ so to show that $\hat{\theta}$ and $\hat{c}$ are Lipschitz on $\mathcal{D}_a$ it remains to show that the partial derivatives are bounded on $\mathcal{D}_{b/r} \cap \mathcal{D}_a$. This is equivalent to taking $s \geq a$ as defined in (8). Taking partial derivatives of $\hat{\theta}$ and $\hat{c}$ and rewriting them in terms of s and J gives

$$
\hat{\theta}_x = -\frac{\mu - r}{\sigma^2}\left[1 - sJ''\right].
$$

From (8) it is clear that sJ''' is bounded for $s \geq a$ since $sJ''' \to 0$ as $s \to \infty$. Hence $\hat{\theta}_x$ is bounded on $\mathcal{D}_a$. Next we have

$$
\hat{\theta}_y = \frac{\mu - r}{\sigma^2}[sJ'' - J'(1 - sJ''')]
$$

and once again using (8) it is straightforward to check that sJ'' and J' are bounded for $s \geq a$ since they also tend to 0 as $s \to \infty$. This implies that $\hat{\theta}_y$

is bounded on $\mathcal{D}_a$. Hence $\hat{\theta}$ in Lipschitz on $\mathcal{D}_a$. Now for $\hat{c}$, we have

$$\hat{c}_x = \begin{cases} 0 & \text{for} \quad a \le s \le 1 \\ \frac{s^{-1-1/R}}{RJ''} & \text{for} \quad 1 \le s \le b^{-R} \\ 0 & \text{for} \quad b^{-R} \le s < \infty \end{cases}$$

$$\hat{c}_y = \begin{cases} 1 & \text{for} \quad a \le s \le 1 \\ \frac{1}{R}\left(Rs^{-1/R} + \frac{s^{-1-1/R}J'}{J''}\right) & \text{for} \quad 1 \le s \le b^{-R} \\ 0 & \text{for} \quad b^{-R} \le s < \infty. \end{cases}$$

We just need to check boundedness of $\hat{c}_x$ and $\hat{c}_y$ on $1 \le s \le b^{-R}$ but this is immediate since $\hat{c}_x$ and $\hat{c}_y$ are continuous functions of s on this compact set so are bounded there. Hence $\hat{c}$ is Lipschitz on $\mathcal{D}_a$ as well. □

Proof of Lemma 2. As mentioned in section 3, proving the existence and uniqueness of a strong solution to (14) and (15) is equivalent to proving the existence and uniqueness of a strong solution to (14) and (16). We will show the latter by adapting the argument used in the proof of Proposition 6.2 in Elie and Touzi [10]. By Lemma 1, we have that $\hat{\theta}$ and $\hat{c}$ are Lipschitz on $\mathcal{D}_a$. Define $\hat{C}(x,y) = rx - \hat{c}(x,y)$. Clearly, $\hat{C}$ is also Lipschitz on $\mathcal{D}_a$. Let K represent a Lipschitz constant for $\hat{C}$. For a fixed $y > 0$ consider the function $G(t,\mathbf{x}) = \hat{C}(\mathbf{x}(t), \max\{y, \bar{\mathbf{x}}(t)/z_a\})$ defined on $\mathbb{R}_+ \times \mathcal{C}_0(\mathbb{R}_+)$ where $\bar{\mathbf{x}}(t) = \sup_{0\le s\le t} \mathbf{x}(s)$. Since $\hat{C}$ is Lipschitz, we have that

$$\begin{aligned} |G(t,\mathbf{x}_1) - G(t,\mathbf{x}_2)| &\le K\{|\mathbf{x}_1(t) - \mathbf{x}_2(t)| + |\max\{y, \bar{\mathbf{x}}_1(t)/z_a\} - \max\{y, \bar{\mathbf{x}}_2(t)/z_a\}| \\ &\le K\left(1 + \frac{1}{z_a}\right) \sup_{0\le s\le t} |\mathbf{x}_1(s) - \mathbf{x}_2(s)|. \end{aligned}$$

This proves that G is functional Lipschitz as defined on page 250 in Protter [20]. A similar argument shows that $\hat{\theta}$ is also functional Lipschitz. Then the existence and uniqueness of a strong solution follows from Theorem 7 page 253 in Protter [20]. □

Proof of Lemma 3. Define the functions $\hat{\delta} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ and $\hat{\pi} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\hat{\pi}(x,y) = \frac{\hat{\theta}(x,y)}{x - \frac{by}{r}}$$

$$\hat{\delta}(x,y) = \frac{\hat{c}(x,y) - by}{x - \frac{by}{r}}$$

for $x/y > b/r$ and set $\hat{\pi}(x,y) = \hat{\delta}(x,y) = 0$ when $x/y = b/r$. Recall that $\hat{\theta}$ and $\hat{c}$ are Lipschitz on $\mathcal{D}_a$ by Lemma 1, and that $\hat{\theta}(x,y) = 0$ and $\hat{c}(x,y) = by$ when $x/y = b/r$. From this it follows that $\hat{\pi}$ and $\hat{\delta}$ are bounded on $\mathcal{D}_a$. Now let (X,Y) be the unique strong solution to (14) and (15). Define $\pi_t^* = \hat{\pi}(X_t, Y_t)$ and $\delta_t^* = \hat{\delta}(X_t, Y_t)$. Then we can rewrite (4) as

$$dX_t^{x,\theta^*,c^*} = \left(X_t^{x,\theta^*,c^*} - \frac{bY_t^{y,c^*}}{r} \right) [(r + \pi_t^*(\mu - r) - \delta_t^*)dt + \sigma\pi_t^* dW_t] .$$

Now we introduce a change of variable similar to that used in Cvitanić and Karatzas [4]. Define

$$\tilde{X}_t^{x,y,\theta^*,c^*} = \left(X_t^{x,\theta^*,c^*} - \frac{bY_t^{y,c^*}}{r} \right) (Y_t^{y,c^*})^{\frac{b}{z_a r - b}} . \tag{31}$$

By Itô's formula,

$$\begin{aligned} d\tilde{X}_t^{x,y,\theta^*,c^*} &= \tilde{X}_t^{x,y,\theta^*,c^*} [(r + \pi_t^*(\mu - r) - \delta_t^*)dt + \sigma\pi_t^* dW_t] \\ &\quad + \left(\frac{b}{ar - b} \right) (Y_t^{y,c^*})^{\frac{b}{ar-b}-1} \left(X_t^{x,\theta^*,c^*} - z_a Y_t^{y,c^*} \right) dY_t^{y,c^*} . \end{aligned}$$

However, the last term vanishes by (16). We can solve this stochastic differential equation explicitly to obtain

$$\begin{aligned} \tilde{X}_t^{x,y,\theta^*,c^*} &= \left(X_0 - \frac{bY_0}{r} \right) Y_0^{\frac{b}{z_a r - b}} \\ &\quad \times \exp\left(\int_0^t \sigma\pi_s^* dW_s + \int_0^t (r + \pi_s^*(\mu - r) - \delta_s^* - \sigma^2(\pi_s^*)^2) ds \right) . \end{aligned}$$

This is non-negative almost surely hence from (31) we have $X_t^{x,\theta^*,c^*}/Y_t^{y,c^*} \geq b/r$ almost surely for all $t \geq 0$. Recall that $c_t^* = \hat{c}(X_t, Y_t)$. From the definition of $\hat{c}$ in (13), we deduce that $c_t^* \geq bY_t^{y,c^*}$ almost surely for all $t \geq 0$. Also, the boundedness of $\hat{\pi}$ and $\hat{\delta}$ together with the continuity of X and Y imply conditions (1) and (2). Putting this all together gives that $(\theta^*, c^*) \in \mathcal{A}_b(x,y)$. □

7 The problem is ill-posed for $R \leq R^*$

In the standard Merton problem [17], one observes that for $R \leq R^*$ (for R^* as defined in (7)), it is possible to find investment and consumption strategies that give infinite expected utility. It therefore comes as no surprise that we observe the same here. This implies that the Merton problem with a drawdown constraint on consumption is ill-posed for $R \leq R^*$. That is, for $R \leq R^*$ it is possible to find investment and consumption strategies such that (†) does not have a finite maximum.

In the previous sections, we presented and verified the optimal solution for $R > R^*$. Now, for completeness, we will construct a class of investment and consumption strategies to show that (†) does not have a finite maximum for $R \leq R^*$. The consumption strategy that we will construct will be non-decreasing (equivalent to taking $b = 1$) thus this will satisfy the drawdown constraint (5) for all $0 \leq b \leq 1$. For convenience, we drop the superscripts on $X_t^{x,\theta,c}$ and $Y_t^{y,c}$.

Proposition 2. *Fix $R \leq R^*$, and take $(X_0, Y_{0-}) = (x, y) \in \mathcal{D}_{1/r}$. Define $(\theta, c) = (\theta_t, c_t)_{t \geq 0}$ by $c_t = \max\{y, \lambda \bar{X}_t\}$ and $\theta_t = \pi_M(X_t - \frac{c_t}{r})$ for $\bar{X}_t = \sup_{0 \leq s \leq t} X_s$ and where $\pi_M = \frac{\mu - r}{\sigma^2 R}$ and $0 < \lambda < \frac{r\kappa^2}{2rR + 2\kappa^2}$. Then under this investment and consumption strategy (θ, c) we have that*

$$\mathbb{E}\left[\int_0^\infty e^{-\rho t} U(c_t) dt\right] = \infty.$$

Proof. Under (θ, c) our wealth equation is

$$dX_t = \left(X_t - \frac{c_t}{r}\right)\left[\left(r + \frac{\kappa^2}{R}\right) dt + \frac{\kappa}{R} dW_t\right]. \tag{32}$$

Using a similar argument as in the proofs of Lemmas 1 and 2 one can show that there exists a unique strong solution to (32). Note that the consumption strategy c is non-decreasing so satisfies the drawdown constraint (5) for all $0 \leq b \leq 1$. Also by continuity of X and $\bar{X}$ we have that conditions (1) and (2) are satisfied. Thus $(\theta, c) \in \mathcal{A}_b(x, y)$ for all $0 \leq b \leq 1$.

Now as we did in the proof of Lemma 3, we modify the change of variable used in Cvitanić and Karatzas [4]. Define

$$\hat{X}_t = \left(X_t - \frac{c_t}{r}\right) c_t^\nu \tag{33}$$

for $\nu = \frac{\lambda}{r - \lambda}$. Then following the same method as in the proof of Lemma 3 we get that

$$\hat{X}_t = \hat{x}_0 \Gamma_t \exp\left[t\left(r + \frac{\kappa^2}{R}\right)\right]$$

where $\hat{x}_0 = (X_0 - c_0)c_0^\nu$ and $\Gamma_t = \exp[\frac{\kappa}{R}W_t - \frac{\kappa^2 t}{2R^2}]$. This implies that

$$\sup_{0\le s\le t} \hat{X}_s = \hat{x}_0 \bar{\Gamma}_t \exp\left[t\left(r + \frac{\kappa^2}{R}\right)\right] \tag{34}$$

where $\bar{\Gamma}_t = \sup_{0\le s\le t}\Gamma_s$. However, note that when c is increasing we have that $c_t = \lambda\bar{X}_t$ thus from (33), we also have

$$\sup_{0\le s\le t} \hat{X}_s = \left(\bar{X}_t - \frac{c_t}{r}\right)c_t^\nu \le \left(\frac{1}{\lambda} - \frac{1}{r}\right)c_t^{1+\nu} \quad \text{a.s.} \tag{35}$$

since $c_t \ge \lambda\bar{X}_t$ almost surely by definition. Equating (34) and (35) gives

$$\begin{aligned} c_t &\ge \tilde{c}_0 \exp\left[t\left(r + \frac{\kappa^2}{R}\right)\left(\frac{1}{1+\nu}\right)\right]\bar{\Gamma}^{\frac{1}{1+\nu}} \\ &\ge \tilde{c}_0 \exp\left[t\left(r + \frac{\kappa^2}{R}\right)\left(\frac{1}{1+\nu}\right)\right] \quad \text{a.s.} \end{aligned}$$

where $\tilde{c}_0 = [\hat{x}_0(\frac{1}{\lambda} - \frac{1}{r})^{-1}]^{\frac{1}{1+\nu}}$ and in the last inequality we used that $\bar{\Gamma}_t^{\frac{1}{1+\nu}} \ge 1$ almost surely. This gives

$$\begin{aligned} &\mathbb{E}\left[\int_0^\infty e^{-\rho t}U(c_t)dt\right] \\ &= \int_0^\infty e^{-\rho t}\mathbb{E}\left[U(c_t)\right]dt \\ &\ge \int_0^\infty U(\tilde{c}_0)\exp\left[t\left(-\rho + (1-R)(1-R)\left(\frac{1}{1+\nu}\right)\left(r + \frac{\kappa^2}{R}\right)\right)\right]dt \\ &= \infty \end{aligned}$$

since the exponent is positive for our choice of λ since $R \le R^*$. □

8 Conclusion

In this paper, we investigate the Merton problem with a drawdown constraint on consumption. We work with CRRA utility and state our candidate value function and candidate optimal controls which are then verified. The key parameter for this problem is the ratio of the investor's wealth to the running maximum of past consumption. Under the optimal control, the amount of wealth in the stock increases non-linearly with this ratio. For the optimal consumption rule, we observe four different regions of behaviour based on the value of this ratio. We also present an argument to show that this problem is ill-posed if the coefficient of relative risk aversion, $R \le R^*$ for a particular $0 < R^* < 1$ which we specify.

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A Heuristic derivation of candidate value function and candidate optimal controls

In this section, we describe the heuristic argument that was used to derive the candidate value function and candidate optimal controls given by (11), (12) and (13). We stress that this argument is heuristic. We will make assumptions as necessary to proceed with the derivation and in some places justification given will be informal. The aim of this argument is just to guess the value function and optimal controls since we provide a rigorous verification argument in section 5.

Essentially, what we will do is derive a set of equations that the true value function should satisfy. Then we will show that subject to appropriate boundary conditions there is only one solution to this set of equations, which we take as our candidate value function.

To start, define the value function

$$V(x,y) = \sup_{(\theta,c)\in\mathcal{A}_b(x,y)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}U(c_t)dt\right]$$

for $(x,y)\in\mathcal{D}_{b/r}$. We make the following assumption.

Assumption 1. We assume that V is twice continuously differentiable in x and continuously differentiable in y with $V_x > 0$ and $V_{xx} < 0$.

Next, as in (21), for $(\theta,c)\in\mathcal{A}_b(x,y)$ define

$$\xi_t^{x,y,\theta,c} = \int_0^t e^{-\rho s}U(c_s)ds + e^{-\rho t}V(X_t^{x,\theta,c}, Y_t^{y,c}).$$

By the Davis–Varaiya Martingale Principle of Optimal Control [7], we should have that $\xi^{x,y,\theta,c}$ is a supermartingale for all $(\theta,c)\in\mathcal{A}_b(x,y)$, and there exist optimal controls (θ^*,c^*) (to be found) such that ξ^{x,y,θ^*,c^*} is a true martingale. By Itô's formula,

$$d\xi_t^{x,y,\theta,c} = \sigma\theta_t V_x dW_t + \mathcal{L}^{\theta,c}V dt + V_y dY_t^{y,c}$$

where

$$\mathcal{L}^{\theta,c}V(x,y) = -\rho V + V_x(rx+\theta(\mu-r)-c) + \frac{1}{2}\sigma^2\theta^2 V_{xx} + U(c).$$

We deduce that we require $V_y \leq 0$ and under the optimal controls (θ^*, c^*) when Y_t^{x,y,θ^*,c^*} is increasing we must have $V_y(X_t^{x,\theta^*,c^*}, Y_t^{y,c^*}) = 0$ almost surely. We also require

$$\sup_{\theta\in\mathbb{R},c\geq by}\left[-\rho V + V_x(rx+\theta(\mu-r)-c)+\frac{1}{2}\sigma^2\theta^2V_{xx}+U(c)\right]=0.$$

Thus we expect the Hamilton-Jacobi-Bellman (HJB) equation for this problem to be

$$\max\left\{V_y, \sup_{\theta\in\mathbb{R},c\geq by}\left[-\rho V + V_x(rx+\theta(\mu-r)-c)+\frac{1}{2}\sigma^2\theta^2V_{xx}+U(c)\right]\right\}=0. \tag{36}$$

Now, because we consider CRRA utility, we can use scaling to reduce to a one-dimensional problem. Define $v(z) = V(z, 1)$ for $z \geq b/r$. We have the following result.

Proposition 3. *$V(x, y) = y^{1-R}V(x/y, 1) = y^{1-R}v(z)$ where we define $z = x/y$.*

Proof. Take $\lambda > 0$. From the linearity of wealth dynamics (4) we have that

$$(\theta, c) \in \mathcal{A}_b(\lambda x, \lambda y) \quad \Leftrightarrow \quad (\tilde{\theta}, \tilde{c}) \in \mathcal{A}_b(x, y)$$

where $(\tilde{\theta}, \tilde{c}) = (\theta/\lambda, c/\lambda)$. Now observe that

$$\begin{aligned}
V(\lambda x, \lambda y) &= \sup_{(\theta,c)\in\mathcal{A}_b(\lambda x,\lambda y)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\frac{c_t^{1-R}}{1-R}\right)dt\right] \\
&= \sup_{(\tilde{\theta},\tilde{c})\in\mathcal{A}_b(x,y)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\frac{(\lambda\tilde{c}_t)^{1-R}}{1-R}\right)dt\right] \\
&= \lambda^{1-R}\sup_{(\tilde{\theta},\tilde{c})\in\mathcal{A}_b(x,y)} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\left(\frac{\tilde{c}_t^{1-R}}{1-R}\right)dt\right] \\
&= \lambda^{1-R}V(x, y).
\end{aligned}$$

Taking $\lambda = 1/y$ gives the result. □

Proposition 3 suggests that the key parameter in this problem is $z = x/y$, or in terms of processes $Z_t^{x,y,\theta,c} = X_t^{x,\theta,c}/Y_t^{y,c}$, the ratio of wealth to the running maximum of past consumption. We guess that there is a type of threshold behaviour which depends on the value of this ratio. To be precise, we assume that the first term in the HJB equation is equal to zero if and only if $z > z_a$ and that the second term is equal to zero if and only if $z \leq z_a$ for some z_a to be determined. The intuitive reasoning for this is that the first term is only zero when we increase Y_t^{y,c^*} under the optimal controls (θ^*, c^*) which would only happen if Z_t^{x,y,θ^*,c^*} were large. This is because large Z_t^{x,y,θ^*,c^*} means that our wealth is very large compared to the running maximum of past consumption, so we have more than enough wealth to maintain consumption at its current maximum, so it is in our best interests to raise Y_t^{y,c^*} and increase consumption from then on.

Consider the region $z \leq z_a$ first, which corresponds to the second term in the HJB equation. We can divide this into two maximisation problems. The first is

$$\sup_{\theta \in \mathbb{R}} \left[\theta(\mu - r)V_x + \frac{1}{2}\sigma^2\theta^2 V_{xx} \right]$$

which is maximised by

$$\hat{\theta}(x, y) = -\frac{\mu - r}{\sigma^2}\frac{V_x(x, y)}{V_{xx}(x, y)} \tag{37}$$

The second maximisation is

$$\sup_{c \geq by}\{U(c) - cV_x\}$$

which is maximised by

$$\hat{c}(x, y) = \begin{cases} by & \text{for} \quad b/r \leq x/y \leq z_{b^{-R}} \\ (V_x(x, y))^{-1/R} & \text{for} \quad z_{b^{-R}} \leq x/y \leq z_1 \\ y & \text{for} \quad z_1 \leq x/y \leq z_a \end{cases} \tag{38}$$

where $z_{b^{-R}} = v'(b^{-R})$ and $z_1 = v'(1)$.

For $z \geq z_a$, recall that we assume a threshold-type behaviour so we guess that the optimal strategy is to immediately increase consumption to a new maximum so that z falls back to the threshold value. In more precise terms, we guess that the optimal strategy is to take

$$\hat{c}(x, y) = \frac{x}{z_a} \quad \text{for} \quad z_a \leq z < \infty. \tag{39}$$

Substituting (37) and (38) into the (36) and using Proposition 3 to write everything in terms of v and z gives

$$\begin{array}{rclll} 0 & = & -\rho v + rzv' - \frac{1}{2}\kappa^2 \frac{(v')^2}{v''} + U(b) - bv' & \text{for} & b/r \leq z \leq z_{b^{-R}} \\ 0 & = & -\rho v + rzv' - \frac{1}{2}\kappa^2 \frac{(v')^2}{v''} + \tilde{U}(v') & \text{for} & z_{b^{-R}} \leq z \leq z_1 \\ 0 & = & -\rho v + rzv' - \frac{1}{2}\kappa^2 \frac{(v')^2}{v''} + U(1) - v' & \text{for} & z_1 \leq z \leq z_a \\ 0 & = & (1-R)v - zv' & \text{for} & z_a \leq z < \infty. \end{array} \tag{40}$$

By Proposition 1, for all $(\theta, c) \in \mathcal{A}_b(x, y)$ we have $(X_t^{x,\theta,c}, Y_t^{y,c}) \in \mathcal{D}_{b/r}$ almost surely for $t \geq 0$. At the boundary where $Z_t^{x,y,\theta,c} = b/r$, by (4) we see that if $\theta_t \neq 0$ then the effect of the dW_t term would mean that with positive probability $Z_t^{x,y,\theta,c}$ would fall below b/r immediately. Thus if $Z_t^{x,y,\theta,c} \downarrow b/r$ we must have $\theta_t \downarrow 0$ almost surely so that all our wealth is in the bank account and to maintain the drawdown constraint we consume the interest $rX_t^{x,\theta,c} = bY_t^{y,c}$ which keeps $Z_t^{x,y,\theta,c}$ constant at b/r. This reasoning holds for the optimal controls as well which gives us two boundary conditions at $z = b/r$. The first is

$$v(b/r) = U(b)/\rho \tag{41}$$

and we also have $\hat{\theta} \downarrow 0$ as $z \downarrow 0$ which by (37) and Proposition 3 is equivalent to

$$\frac{v'(z)}{v''(z)} \downarrow 0 \quad \text{as } z \downarrow b/r. \tag{42}$$

To solve (40) subject to these boundary conditions we transform to dual variables. Define

$$J(s) = \sup_{z > b/r} \{v(z) - sz\}. \tag{43}$$

By Assumption 1 and Proposition 1, we have that $v' > 0$ and $v'' < 0$ so the supremum in (43) is attained for $z = (v')^{-1}(s)$. Also since v is twice continuously differentiable, so is J. Using the link between the derivatives of v and its dual J (we refer the reader to Rockafellar [22] for more details on dual transforms) we can rewrite (40) as

$$\begin{array}{rclll} 0 & = & (1-R)J + RsJ' & \text{for} & 0 < s \leq a \\ 0 & = & -\rho J + (\rho - r)sJ' + \frac{1}{2}\kappa^2 s^2 J'' + U(1) - s & \text{for} & a \leq s \leq 1 \\ 0 & = & -\rho J + (\rho - r)sJ' + \frac{1}{2}\kappa^2 s^2 J'' + \tilde{U}(s) & \text{for} & 1 \leq s \leq b^{-R} \\ 0 & = & -\rho J + (\rho - r)sJ' + \frac{1}{2}\kappa^2 s^2 J'' + U(b) - bs & \text{for} & b^{-R} \leq s \leq s_{b/r} \end{array} \tag{44}$$

where $s_{b/r} = v'(b/r)$. We guess that $s_{b/r} = \infty$ which makes intuitive sense because if Z_t^{x,y,θ^*,c^*} ever hits b/r then by the above reasoning Z_t^{x,y,θ^*,c^*} is held constant at this value and we have no choice but to consume at the minimum allowed level from this point onwards. Thus any deviation from this point would be significantly more preferable than remaining there, which would suggest that $v'(b/r) = \infty$.

Rewriting the boundary conditions (41) and (42) in terms of J and s gives

$$sJ''(s) \to 0 \quad \text{as } s \to \infty \tag{45}$$

and

$$\left| J(s) - \frac{U(b)}{\rho} + \frac{b}{r} s \right| \to 0 \quad \text{as } s \to \infty. \tag{46}$$

The boundary conditions (45) and (46) together with the continuity of J,J' and J'' imply that there is a unique solution to (44) which is given by (8).

We can recover v by taking the dual of J (see Rockafellar [22] for more details)

$$v(z) = \inf_{0<s<\infty} \{J(s) + zs\}$$

and taking $V(x, y) = y^{1-R} v(x/y)$ gives our candidate value function as given in (11).

Our candidate investment and consumption strategy is given by (37), (38) and (39).