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# ON STRATEGIC DEFAULT
# AND LIQUIDITY RISK

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# ON STRATEGIC DEFAULT AND LIQUIDITY RISK

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**Abstract:** How does the uncertain provision of external finance affect investment projects' default probability and liquidity risk? In this paper I study the strategic interaction between many creditors and a single borrower in the context of a two-period investment project requiring external credit. Loans mature in one period but the project requires two periods to complete. The key working assumptions are that creditors are risk-averse and that any uncertainty is common knowledge: information about the fundamentals can be incomplete but not asymmetric. Mixed and perfect Bayesian strategies are used to compute the equilibrium probabilities of default and early liquidation. The impact of the maturity structure on default and liquidity risk is a function of the underlying structural and stochastic parameters and investors' beliefs about the state of fundamentals. The implications for banking regulation are assessed under fixed and variable loan rates. An open range of fundamentals is derived outside of which default and liquidity risk are either zero or one. The cyclical properties of default and liquidity risk are shown to depend sensitively on the relative cost of early liquidation to the borrower and the creditors, hence also on the regulatory policy stance.

**JEL Classification:** C72, F32, F34

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# 1  Introduction

How does the uncertain provision of external funding affect investment project outcomes? The recent financial crises in Asia, Russia and Latin America and the resulting sharp declines in real output have focussed research attention on the role of the available amount of liquidity—and the lack of it—in bringing about crises which become self-fulfilling. The two main types of crises models are fundamental-based and belief-based. The first category starts with Krugman (1979), in which a deterministically deteriorating current account results in devaluation, while the second is associated with the work of Obstfeld (1996,1997). Morris and Shin (1998) have shown that beliefs-based currency crises can yield a unique equilibrium in the presence of asymmetric information about the fundamentals.[1] More generally, liquidity is determined at the level of microeconomic decision-makers (individual creditors/investors) and translated into aggregate liquidity. Holmstrom and Tirole (1996, 1998) show that if markets are incomplete then entrepreneurs may be unable to insure themselves against exogenous shocks to net worth. Investment projects that are socially valuable may thus be prematurely terminated.

In this paper, I study the dynamic provision of liquidity to an investment project requiring external funding as a two-period game between many risk-averse lenders and a single borrower. Unlike Bulow and Rogoff (1989a,b) and Mella-Barral and Perraudin (1997), the debt is privately held and there is no possibility for strategic recontracting and rescheduling. The project has constant returns to scale with respect to the internal and external funding obtained. Investment performance is a linear function of a shock to macroeconomic fundamentals which is only realized after lending has been committed. At the outset, outside investors decide how much to lend based on their individual risk aversion. It is assumed that the maturity of the loan

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[1]For a survey of the fundamental and belief-based crisis literature see Chui et al. (2000).

is one period, but the investment project requires two periods to complete. The borrower cannot choose the maturity structure: at the end of the first period, investors may decide not to roll over their loan and instead withdraw. Early withdrawal attracts a penalty which is increasing in the amount of individual investment, but bounded by limited liability. The disruption to the project brought about by early liquidation is also linearly increasing in the amount of credit withdrawn. Faced with a balance sheet structured with a long-term asset and short-term external liabilities, the borrower's position is therefore subject to liquidity and interest rate risk.

There are three stages in the analysis. In the first, the loan rate is fixed and the game is sequential. Information about the fundamentals is then complete and perfect. I obtain Nash equilibria in pure dominant strategies and show the existence of a range of fundamentals below which all lenders withdraw early, and above which they all roll over into the second period. The properties of this range are characterised as a function of the underlying parameters. These include the project's loan rate, the riskless (world) interest rate, the penalty for early withdrawal and the disruption it causes the project, the contribution to project finance of the borrower's internal endowment, and the probability distributions of the fundamentals and lenders' risk aversion.

In the second stage, the loan rate is fixed and the game is one of complete but imperfect information. The project's default probability and the degree of early liquidation are then determined endogenously as part of mixed strategies Nash equilibrium. In the third stage, I introduce a variable loan rate and incomplete information about the fundamentals. Importantly, however, information is symmetric: the lenders and borrower all observe the fundamental realisation, but are uncertain of the true underlying state of the world, good or bad. All risk is, therefore, *systematic*. This is founded on the premise that, in globalised markets, extrinsic uncertainty—that is, lack of common knowledge—matters as much as, if not more than, the asymmetric

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information channel. This premise contrasts the seminal work of Diamond and Dybvig (1983), where lenders refuse to finance an illiquid borrower because of strategic uncertainty about other lenders' actions.[2]

A rational set of prior beliefs and Bayesian updates is developed which leads to a perfect Bayesian equilibrium characterisation. The degree of risk aversion and the strategic interaction of lenders and borrower turn out to have strong implications for aggregate liquidity provision. The model provides a mechanism for assessing recent alternative proposals for international financial regulation reform aimed at preventing financial crises.[3] In particular, the desirability of imposing restrictions on short-term capital flows can be analysed via the impact on the project's default probability of varying the penalty charged to creditors for liquidating early. In turn, the impact of this penalty on liquidity risk is a function of the disruption that early withdrawal causes the investment project. The disruption is assumed to be increasing in the amount of outside credit, reflecting the project's sensitivity to short-term reversals of investor sentiment.

The main findings can be summarised as follows. Assuming a uniform fundamental distribution, equilibrium liquidity risk is found to be pro-cyclical, that is positivelty correlated with the business cycle. Consequently, there is a smaller likelihood of capital flow reversals in periods of macroeconomic slowdown. Strategic liquidity provision can thus be said to exert a stabilising influence over the business cycle. However, the concerns raised by the Bank of International Settlements' proposals for reforming capital adequacy ratios centre on the likelihood of pro-cyclical credit quality and countercyclical aggregate default risk, which are clearly destabilising.[4] In that respect, I find that for the default probability to be pro-cyclical—that is, for default risk to

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[2]See also Diamond (1991), where borrowers can choose the optimal debt maturity.

[3]For example, see Eatwell and Taylor (2000) and Eichengreen et al. (1995).

[4]See the Basel Committee on Banking Supervision (2001) and Borio et al. (2001).

fall during recessions and rise during expansions—the disruption caused by early liquidation has to be small. In that case, international financial regulators aiming to maintain a stable investment environment should impose fewer restrictions on short-term capital flows in times of expansion and more during recession. In contrast, if the disruption caused by early liquidation is large then aggregate default risk becomes counter-cyclical. Regulators should then be imposing fewer short-term capital controls during recessions.

The remainder of the paper is arranged as follows: Section 2 presents the model; Section 3 derives the lenders' and borrower's optimal strategies when information is complete and the loan rate is exogenously fixed. When the game is sequential, the solution involves pure dominant strategies; when it is simultaneous, the solution concept is mixed strategies Nash equilibrium. Section 4 studies the case of incomplete information when the loan rate is a function of the fundamental realisation; Section 5 concludes the paper.

## 2 The model

### 2.1 Investment technology and timing

There is a single domestic borrower/entrepreneur and a large number of atomistic risk-averse investors (outside creditors). The investment project is not self-financing and requires two periods to complete. Gross project income at the end of period 2 is given by:

$$y = \theta(E + L) \tag{1}$$

$E$ denotes the amount of internal illiquid endowments available to the project, $L$ is the amount of outside lending, and $\theta$ is a random productivity shock which is positively correlated with the state of domestic macroeconomic fundamentals. In the context of foreign direct investment, $L$ can be

interpreted as the amount of project finance which is obtained abroad.

The fundamental realisation $\theta$ is drawn from a uniform distribution over the closed interval $[0, 1]$ and is assumed to be perfectly observable by the borrower and lenders. The borrower can also access liquid reserve assets $A$. These yield the riskless rate of return $r_A$, which coincides with the (fixed) world rate of interest. Initially, outside credit is obtained for two periods. However, at the end of the first period a proportion $\lambda \in [0, 1]$ of lenders may decide not to roll over their loans into the second period. In that case, the borrower's liquid assets $A$ can be used to cover the resulting liquidity shortfall. Therefore, the magnitude of $\lambda$ represents liquidity risk, measured as the probability that the borrower will lose the project's rents due to excessive liquidation incentives of lenders.

The early withdrawal of funding induces: (i) A marginal cost $k < 1$ to the investment project, borne by the borrower. The parameter $k$ captures the marginal project disruption brought about by early liquidation. The total disruption cannot exceed the aggregate amount of lending. (ii) A marginal loss $c < 1$ on the individual loan amount, borne by each lender. This parameter can be considered to be negatively correlated with the amount of short-term capital controls in place, and positively correlated with the cost of enforcing repayment. The values of $k$ and $c$ could, in principle, both be influenced by the domestic and international regulatory framework.

The borrower thus has to satisfy the following linear constraint on the project's net return:

$$\theta(E + L) - k\lambda L + (1 + r_A)(A - \lambda L) \geq (1 - \lambda)(1 + r_L)L \qquad (2)$$

On the LHS is the borrower's return; the value of the investment project declines in the amount of early liquidation $\lambda L$. The decline is monotonically increasing in the marginal cost of disruption $k$ and in the proportion of lenders $\lambda$ who do not roll over their loan in period 1. Some of the borrower's

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liquid assets $A$ then have to be used to cover the shortfall from liquidating lenders. On the RHS is the total payment to lenders; the borrower's cash outflow is increasing linearly in the rate of return (loan rate) $r_L$ offered to the proportion $1 - \lambda$ of lenders who stay on for the maturity of the project. Loans rate can be fixed (Section 3) or monotonically decreasing in the state of fundamentals (Section 4), thereby introducing interest rate (market) risk.

## 2.2 Strategies and payoffs

A total amount of outside funding $L$ has been committed at the start of the game $(t = 0)$ for one or two periods. In the first period $(t = 1)$, individual lender $i$ can *Stay*, with probability $\lambda$, or *Leave* $(1 - \lambda)$. The borrower can *Default* on the project with probability $P$ or *Repay* $(1 - P)$, either at $t = 2$ (Section 3.1) or simultaneously with the lenders (Section 3.2). The final payoffs of the game are represented in strategic form in Table 1. The strategies of lender $i$ and the single borrower are in rows and columns. Their respective payoffs are given by the top and bottom entries for each strategy combination:[5]

<div align="center">

**Table 1**

**The strategic form game**

</div>

|  | *Repay* $[1 - P]$ | *Default* $[P]$ |
|---|---|---|
| *Leave* | $L_i(1 - c)$ | $L_i(1 - c)$ |
| $\lambda$ | $\theta(E + L) - kL + (1 + r_A)(A - L)$ | $(1 + r_A)(A - L)$ |
| | | |
| *Stay* | $L_i(1 + r_L)$ | $0$ |
| $1 - \lambda$ | $\theta(E + L) + (1 + r_A)A - (1 + r_L)L$ | $(1 + r_A)A$ |

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[5]Lender $i$'s payoff are a function of their individual loan while the borrower's payoffs involve the aggregate amount of funding.

In pure strategies, all lenders *Stay* ($\lambda = 0$) or *Leave* ($\lambda = 1$), while the single borrower *Defaults* ($P = 1$) or *Repays* ($P = 0$) with certainty. In mixed strategies, the probability $\lambda$ then measures the proportion of atomistic lenders opting to liquidate their investment at $t = 1$. In contrast, the project's default probability $P$ reflects a strategic decision by the borrower independently of any information held by the lenders regarding their true type. In principle, the borrower can default even if no creditor has withdrawn early. Therefore, $P$ and $\lambda$ can, respectively, be interpreted as default and liquidity risk measures.

## 2.3   The optimal lending decision

The return process for the investment project follows a Bernoulli distribution with success defined as repayment, with probability $1 - P$, and failure defined as default, with probability $P$. The first two moments of lender $i$'s payoff if she rolls over her loan at $t = 1$ are:

$$E(x^i) = (1 - P)(1 + r_L)L_i$$

$$(3)$$

$$\sigma^2(x^i) = P(1 - P)(1 + r_L)^2 L_i^2$$

Moreover, lenders cannot expect to receive less than the constant riskless rate of return $r_A$. This implies the following weak participation constraint:

$$Ex_i \geq (1 + r_A)L_i \iff 1 + r_L \geq \frac{1 + r_A}{1 - P} \qquad (4)$$

Inequality (4) guarantees that, in expectation, the lender obtains at least the riskless rate of return. Given the value of $r_A$, the credit spread $r_L - r_A$ is always positive and increasing in the default probability.

The amount of funding provided by lender $i$ at $t = 0$ is determined by simple quadratic optimisation. In the absence of strategic considerations, lender $i$ maximises the quadratic utility function:

$$U^i = E(x^i) - b_i \sigma^2(x^i) \tag{5}$$

Lenders are differentiated according to their risk aversion coefficient $b_i$. I assume $b_i$ to be a random variable uniformly distributed on the closed interval $[b_{min}, b^{MAX}] = [\underline{b}, 1]$ with $\underline{b} \geq 0$, wlog. The minimum lower bound $\underline{b} = 0$ corresponds to risk neutrality. Given the upper bound $b^{MAX} = 1$, higher $\underline{b}$ values reflect higher aggregate risk aversion. Substituting equations (3) into (5) and maximising with respect to $L_i$ yields the amount of funding that lender $i$ provides at $t = 0$:

$$L_i^* = \frac{1}{2b_i(1 + r_L)P} \tag{6}$$

The optimal amount of funding is decreasing in the loan rate, lender $i$'s risk aversion coefficient and the default probability. Aggregating the individual loan amounts in (6) over the interval $[\underline{b}, 1]$ yields the aggregate outside funding obtained at $t = 0$:

$$L^* = \int_{\underline{b}}^{1} L_i^* db_i = -\frac{\ln \underline{b}}{2(1 + r_L)P} > 0 \tag{7}$$

$L^*$ is decreasing in lenders' aggregate risk aversion. *Ceteris paribus*, larger $\underline{b}$ values result in less of the investment project being externally financed.

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# 3 Equilibrium liquidity and default risk with fixed loan rates

## 3.1 The sequential game with perfect information

I begin by analysing the two-period game with sequential play. First, at $t = 1$ the lenders decide whether to *Stay* or *Leave*. The borrower observes their move and decides to *Repay* or *Default* at $t = 2$ (Figure 1, Panel A). When does lender $i$ have an incentive to *Stay* on to period 2? From the payoff matrix in Table 1, the expected return from rolling over her loan at $t = 1$ should not be less than the certain return from early withdrawal:

$$(1 - P)(1 + r_L)L_i \ \geq \ (1 - c)L_i$$

This yields a threshold level $P^*$ for the default probability:[6]

$$P \ \leq \ P^* \ = \ \frac{c + r_L}{1 + r_L} \tag{8}$$

For the probability range $P \leq P^*$, *Staying* is weakly dominant, implying zero liquidity risk: $\lambda = 0$. Conversely, for the range $P > P^*$ all lenders *Leave*, implying $\lambda = 1$. Note that $P^* < 1$ because $c < 1$ by investors' limited liability. The threshold level $P^*$ is strictly increasing in $c$ and $r_L$. Equivalently, the default probability range $P \in (P^*, 1]$ is smaller for higher $c$ and $r_L$. Imposing a greater penalty to foreign investors for not rolling over their loans and/or offering them higher loan rates makes early withdrawal suboptimal, *ceteris paribus*.

The borrower's best response in period 2 thus depends on the default probability $P$. There are two cases. First, if $P \ \leq \ P^*$ then $\lambda = 0$ and their best response at $t = 2$ is to complete the project and *Repay* if and only if:

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[6]If $P = P^*$ then lender $i$ is indifferent between her two pure strategies.

$$\theta_2(E + L) + (1 + r_A)A - (1 + r_L)L > (1 + r_A)A$$

This implies a threshold level of fundamentals $\theta_2^{min}$ as a function of the lending rate $r_L$, the project's internal endowment $E$ and aggregate lending $L$ above which the borrower will *Repay* the creditors:

$$\theta_2 > \underline{\theta}_2 = \frac{(1 + r_L)L}{E + L} \qquad (9)$$

If the fundamental realisation is less than $\underline{\theta}_2$ then the borrower will default even if all outside creditors decide to roll over their investment at $t = 1$. The threshold level increases in $r_L$ and $L$ and decreases in $E$. Intuitively, higher lending rates and/or smaller internal endowments make default more likely. In contrast, the positive contribution of the amount of aggregate lending to outright default reflects a moral hazard problem. The fact that no lender has left by the end of the first period ($\lambda = 0$) improves the project's expected return and presents the borrower with a greater incentive to *Default*. Equivalently, the range of fundamentals for which the borrower has an incentive to *Repay* is smaller.

However, the default probability threshold $P^*$ in equation (8) need not be consistent with the borrower's best response. Provided $P \leq P^*$, from (9) the borrower will always default if $\underline{\theta}_2 > 1$. Then $(1 + r_L)L > E + L$, equivalently $L > \frac{E}{r_L}$, and the unique Nash equilibrium in pure strategies is $\{Stay, Default\}$. Is this Pareto-inferior outcome a subgame-perfect equilibrium of the extensive form game? In the subgame commencing at $t = 0$, if lenders know that the borrower will certainly default, they will never *Stay* because $0 < L_i(1 - c)$ for all $L_i > 0$. Indeed, in that case the project will obtain no outside credit at $t = 0$. Equivalently, $P^*$ is strictly less than 1, so it is inconsistent with certain default. Therefore, $\{Stay, Default\}$ is not a subgame-perfect Nash equilibrium.

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In the second case, if $P > P^*$ then the default probability exceeds the threshold and not rolling over at $t = 1$ (*Leaving*) becomes the lenders' strictly dominant strategy: $\lambda = 1$. Because lenders enjoy first-mover advantage, *Defaulting* at $t = 2$ may be the borrower's dominant strategy. Therefore, the unique Nash equilibrium in pure dominant strategies is $\{Leave, Default\}$ if and only if:

$$(1 + r_A)(A - L) \;>\; \theta_2(E + L) - kL + (1 + r_A)(A - L)$$

The borrower will default if $\theta_2$ is less than the following upper bound:

$$\theta_2 \;<\; \overline{\theta}_2 \;=\; \frac{kL}{E + L} \tag{10}$$

$\overline{\theta}_2$ is increasing in the disruption caused by early liquidation $(k)$ and in aggregate outside lending $(L)$, and decreasing in the project's internal endowment $(E)$. The intuition is that early liquidation lowers the project's *ex post* return, thus encouraging default. Conversely, more internal funding improves the project's chances of completion following early liquidation, all other things equal. Moreover, $\{Leave, Default\}$ is a subgame-perfect Nash equilibrium because $P > P^*$ is consistent with a default probability of one. Finally, substituting the aggregate outside loan amount from equation (7) into (10) it is easy to verify that higher $P$ yields a wider range of fundamentals for which *Default* is optimal for the borrower.

To summarise, in the sequential game with perfect information default risk measured by $P$ is exogenous. From the creditors' point of view, the dominant strategy then involves a threshold level of $P$ above which no lender will roll over their loan in period 1. From equations (9) and (10), in order for creditors' perception of $P$ to be consistent with the borrower's best response, it has to be that $\underline{\theta}_2 < \theta_2$ if $P < P^*$ (good fundamentals), and $\theta_2 < \overline{\theta}_2$ if $P > P^*$ (bad fundamentals).

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## 3.2 The imperfect information game

I now proceed to derive the mixed strategies Nash equilibrium and obtain interior solutions for $\lambda$ and $P$ in a game where the lenders and borrower decide simultaneously on their respective strategies. Equivalently, using the Harsanyi (1967) transformation, this can be viewed as a sequential game of complete but *imperfect* information. Provided the borrower cannot observe the lenders' action, it is analytically possible to accommodate both timings. The extensive form of this game is shown in Figure 1, Panel B. Let $E\pi^{L_i}$ and $E\pi^B$ denote the expected payoff functions of lender $i$ and the borrower from Table 1:

$$
\begin{aligned}
E\pi^{L_i} &= \lambda[(1-P)L_i(1-c) + PL_i(1-c)] + (1-\lambda)[(1-P)(1+r_L)L_i] \\
&= \lambda L_i[1 - c - (1+r_L)(1-P)] + L_i(1+r_L)(1-P) \quad\quad (11)
\end{aligned}
$$

$$
\begin{aligned}
E\pi^B &= (1-P)\lambda\left(\theta(E+L) - kL + (1+r_A)(A-L)\right) \;+ \\
&\quad (1-P)(1-\lambda)\left(\theta(E+L) + (1+r_A)A - (1+r_L)L\right) \;+ \\
&\quad P\left[\lambda(1+r_A)(A-L) + (1-\lambda)(1+r_A)A\right] \quad\quad (12)
\end{aligned}
$$

In mixed strategies Nash equilibrium, the probability weights that each player assigns to their respective pure strategies are determined by the other player's expected payoff maximisation. Optimising first-order condition (11) with respect to $\lambda$ yields the equilibrium default probability:

$$
\frac{\partial E\pi^{L_i}}{\partial \lambda} = 0 \quad \Rightarrow \quad P^* = \frac{c+r_L}{1+r_L} \quad\quad (13)
$$

Note that equation (13) is a generalisation of dominant strategy condition (8). Existence of an interior solution requires $P^*$ to be a completely mixed strategy: $P^* \in (0,1)$ if $c < 1$. This is normally guaranteed by limited

liability: individual creditors cannot lose more than the amount $L_i$ that they invested in the project. If $P < P^*$, then all creditors will stay on to $t = 2$ ($\lambda = 0$); while if $P > P^*$, they will all liquidate early ($\lambda = 1$). Note that $\frac{\partial P^*}{\partial r_L} = \frac{1-c}{(1+r_L)^2} \geq 0$: the default probability is non-decreasing in $r_L$.

Substituting $P^*$ from equation (13) into (7) yields the following closed-form solution for equilibrium aggregate lending as a function of underlying parameters $\underline{b}$, $c$ and $r_L$:

$$L^* = -\frac{\ln \underline{b}}{2(c + r_L)} > 0 \tag{14}$$

The comparative statics of aggregate lending are as follows:

(1) $L^*$ is decreasing in lenders' risk aversion as captured in $\underline{b} < 1$, the risk aversion coefficient's lower bound. This follows directly from the lenders' quadratic utility functions.

(2) $L^*$ is decreasing in the marginal cost of early liquidation. If short-term capital controls—proxied by $c$—are imposed on outside investors then aggregate lending will decrease, and *vice versa* if capital controls are lifted.

(3) Aggregate lending decreases monotonically in $r_L$. Thus, given any internal endowment level $E$, higher loan rates imply that a larger percentage of the project's finance is funded internally. This is a straightforward consequence of lenders' positive risk aversion.

Turning to the equilibrium percentage of investors who withdraw early, optimising first-order condition (12) with respect to $P$ yields:

$$\frac{\partial E \pi^B}{\partial P} = 0 \quad \Rightarrow \quad \lambda^* = \frac{\theta(E + L) - (1 + r_L)L}{(k - 1 - r_L)L} \tag{15}$$

The parameters determining $\lambda^*$ in equation (15) are those involving the borrower: $c$ does not enter. This reflects the fact that the lenders' optimal mixed strategy is obtained from the borrower's expected maximisation. The comparative statics of liquidity risk are as follows:

(1) $\frac{\partial \lambda^*}{\partial k} < 0$ and $\frac{\partial \lambda^*}{\partial E} < 0$: the percentage of investors not rolling over at $t = 1$ is decreasing in the project disruption due to early withdrawal and in the amount of internal endowment.

(2) The impact of the fundamental realisation is given by $\frac{\partial \lambda^*}{\partial \theta} = \frac{E+L}{(k-1-r_L)L}$, which is always negative. Better fundamentals induce more lenders to roll over their loans at $t = 1$, thus lowering liquidity risk.

(3) The impact on $\lambda^*$ of aggregate lending is just $\frac{\partial \lambda^*}{\partial L} = \frac{E}{(1+r_L-k)\theta}$, which is always positive. More lending induces a larger percentage of lenders to *Stay*. Therefore, if aggregate lending is taken to be pro-cyclical, then the liquidity risk measure is also. The cyclicality of the equilibrium measures is discussed in more detail in Section 4. Note also the business cycle has a greater impact on liquidity risk for lower $\theta$ values. Intuitively, deteriorating fundamentals make outside investors more sensitive to aggregate lending fluctuations.

I now derive sufficient conditions for an interior solution for liquidity risk: $\lambda^* \in (0,1)$. First, in order for $\lambda^* > 0$ the expressions in the numerator and the denominator of (15) must have the same sign. On the one hand, the denominator $(k - 1 - r_L)L$ is always negative because $k < 1$ by definition: the marginal project disruption due to early liquidation cannot exceed one, but the loan rate cannot be less than zero. On the other hand, the numerator is negative (positive) if $L > (<) \frac{\theta E}{1+r_L-\theta}$. Focussing on the negative case, aggregate external lending has to exceed a certain level which is decreasing in the loan rate $r_L$. This corresponds to the fundamental range $\theta < \theta_{MAX} \equiv \frac{(1+r_L)L}{E+L}$. In other words, if the fundamental realisation is less than the threshold level $\theta_{MAX}$ then the corner solution $\lambda^* = 0$ arises and all outside creditors optimally roll over their loans.

Second, the inequality constraint ensuring that $\lambda^* < 1$ is just:

$$\theta(E + L) - (1 + r_L)L > L(k - 1 - r_L) \iff L < \frac{\theta E}{k - \theta}$$

Rearranging the last expression yields $\theta > \theta_{min} = \frac{kL}{E+L}$. Fundamental realisations below the $\theta_{min}$ threshold will yield the corner solution $\lambda^* = 1$ and all lenders will *Leave*. Therefore, combined with the range for which $\lambda^* > 0$, the fundamental range $\theta \in (\theta_{min}, \theta_{MAX})$ supporting a solution $\lambda^* \in (0, 1)$ is:

$$\left( \theta_{min} = \frac{kL}{E+L} \ , \ \theta_{MAX} = \frac{(1+r_L)L}{E+L} \right) \tag{16}$$

Note that $\theta_{min} < \theta_{MAX}$ requires $k < 1 + r_L$, which is always true. The fundamental range over which the players' equilibrium strategies are completely mixed is thus well-defined.

# 4 Equilibrium liquidity and default risk with variable loan rates

## 4.1 Belief specification and Bayesian updating

The assumption that the loan rate is exogenously fixed is now relaxed and $r_L$ made a function of the fundamental realisation. At the beginning $(t = 0)$, Nature selects one of two possible fundamental states $s \in \{G, B\}$. Figure 1, Panel C shows the extensive form of the two-period game. *Good* realisations are drawn from a continuous uniform probability distribution function (pdf) for $\theta$ over $[\theta^G, 1]$. In contrast, the *Bad* state is generated from a continuous uniform pdf for $\theta$ over $[\theta^B, 1]$. It is assumed that $0 < \theta^B < \theta^G$: fundamentals under the *Bad* state can get worse than under the *Good* state. Information is incomplete and imperfect: the true state of fundamentals is unknown to the players, so the extensive form involves two initial nodes corresponding to the alternative fundamental distributions. The first two unconditional moments of the two pdf's are just:

$$E(\theta \mid B) = \frac{1 + \theta^B}{2} \quad < \quad E(\theta \mid G) = \frac{1 + \theta^G}{2}$$

$$\sigma^2(\theta \mid B) = \frac{(1 - \theta^B)^2}{12} \quad > \quad \sigma^2(\theta \mid G) = \frac{(1 - \theta^G)^2}{12}$$

The *Good* state stochastically strictly dominates the *Bad* state because it yields higher expected return and lower risk. The loan terms offered to the creditors now depend on the functional relationship between $r_L$ and $\theta$. The borrower is assumed to observe the fundamental realisation at $t = 1$ and then set the loan rate according to the following monotonically decreasing function of $\theta_1$:

$$1 + r_L(\theta_1) \equiv r_A + \frac{1}{\theta_1} \tag{17}$$

Definition (17) implies that the maximum loan rate is bounded by the lower bound of the fundamental distribution's support in each state of the world $s \in \{G, B\}$. By definition, $r_L^{MAX} = r_A + \frac{1}{\theta^s} - 1$ is higher under the *Bad* state. The minimum loan rate is $r_L^{min} = r_A$ in both states by the common upper bound (unity) of the fundamental distributions' support.

Importantly, although the bounds of each distribution's support and the fundamental realisation $\theta_1$ are common knowledge, the true state of fundamentals is never observed by the players. Let their prior (unconditional) beliefs at $t = 0$ about the underlying state be given by $P_0(G)$ and $P_0(B)$, where $P_0(G) + P_0(B) = 1$. These are assumed to be common to the lenders and borrower.[7] Updating of the prior beliefs about the fundamental state is carried out using the fundamental realisation's impact on the loan rate from equation (17). The posterior (conditional) probabilities of the *Good* and *Bad* states are defined using Bayes' rule:

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[7]Specifying different prior beliefs for the borrower and the creditors is arguably more realistic but would require a more complicated model.

$$P_1(G \mid r_L) \;=\; \frac{P_1(r_L \mid G)P_0(G)}{P_1(r_L \mid G)P_0(G) + P_1(r_L \mid B)P_0(B)}$$

$$P_1(B \mid r_L) \;=\; \frac{P_1(r_L \mid B)P_0(B)}{P_1(r_L \mid G)P_0(G) + P_1(r_L \mid B)P_0(B)}$$

(18)

The conditional probability of observing $r_L$ in state $s \in \{G, B\}$ is specified as the fundamentals support which yields *at least* that loan rate:

$$P_1(r_L \mid s) \;\equiv\; \int_{\theta^s}^{\frac{1}{1+r_L-r_A}} \frac{d\theta}{1-\theta^s}$$

(19)

From equation (19), the two conditional probabilities are just:

$$P_1(r_L \mid G) = \frac{1 - (1 + r_L - r_A)\theta^G}{(1 - \theta^G)(1 + r_L - r_A)} \;,\; P_1(r_L \mid B) = \frac{1 - (1 + r_L - r_A)\theta^B}{(1 - \theta^B)(1 + r_L - r_A)}$$

Substituting these in equations (18) yields the posterior probabilities of the *Good* and *Bad* fundamental states, where $P_1(G \mid r_L) + P_1(B \mid r_L) = 1$:

$$P_1(G \mid r_L) \;=\; \frac{[1 - \theta^G(1 + r_L - r_A)]P_0(G)(1 - \theta^B)}{\sum_{s=G,B}[1 - \theta^s(1 + r_L - r_A)]P_0(s)(1 - \theta^s)}$$

$$P_1(B \mid r_L) \;=\; \frac{[1 - \theta^B(1 + r_L - r_A)]P_0(B)(1 - \theta^G)}{\sum_{s=G,B}[1 - \theta^s(1 + r_L - r_A)]P_0(s)(1 - \theta^s)}$$

(20)

When are these Bayesian posterior beliefs rational? A necessary condition is that Bayesian updating should not contradict the known properties of the underlying fundamental distribution. Intuitively, Bayesian updating of beliefs is rational—or *consistent*—if observing a higher (lower) $\theta$ realisation generates a higher posterior probability of the *Good* (*Bad*) fundamentals state:

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$$\frac{\partial P_1(G \mid r_L)}{\partial \theta} > 0 \quad , \quad \frac{\partial P_1(B \mid r_L)}{\partial \theta} < 0 \tag{21}$$

It can be shown that the posterior beliefs defined by equations (20) satisfy (21) provided $\theta^G > \theta^B$; that is, provided the *Good* state stochastically dominates the *Bad* state.[8] Therefore, the proposed Bayesian updating mechanism offers a good description of the players' learning about the unobserved state of fundamentals. The resulting combination of Nash strategies and rational beliefs is then a perfect Bayesian equilibrium of the extensive form game.[9]

## 4.2 Implications for the cyclicality of default risk

I now study the impact of varying $r_L(\theta)$ on the equilibrium probabilities of early withdrawal (liquidity risk) and project default. The unconditional (prior) expectation of $\theta$ at $t = 0$ and the conditional (posterior) expected value at $t = 1$ are given by:

$$\hat{\theta}_0 = E_0(\theta) = P(G)\left(\frac{1 + \theta^G}{2}\right) + P(B)\left(\frac{1 + \theta^B}{2}\right) \tag{22}$$

$$\hat{\theta}_1 = E_1(\theta \mid r_L) = P(G \mid r_L)\left(\frac{1 + \theta^G}{2}\right) + P(B \mid r_L)\left(\frac{1 + \theta^B}{2}\right) \tag{23}$$

At $t = 0$, lenders use their unconditional expectation of fundamentals $\hat{\theta}_0$ to determine their optimal lending according to equation (6). At $t = 1$, the borrower and lenders use the fundamental realisation to compute the updated expected value $\hat{\theta}_1$ and determine their optimal mixed strategies. Substituting $\hat{\theta}_1$ into equations (13) and (15) yields:

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[8] The algebraic derivation is available from the author upon request.

[9] A further restriction on the limits of rational beliefs would satisfy the sequential equilibrium refinement of Kreps and Wilson (1982).

$$P^* = \frac{\hat{\theta}_1(c + r_A - 1) + 1}{1 + r_A \hat{\theta}_1} \tag{24}$$

$$\lambda^* = \frac{\hat{\theta}_1^2(E + L) - (1 + r_A\hat{\theta}_1)L}{[(k - r_A)\hat{\theta}_1 - 1]L} \tag{25}$$

The equilibrium default probability is affected as follows. Equation (24) implies $\frac{\partial P^*}{\partial \hat{\theta}_1} = \frac{c-1}{(1+r_A\hat{\theta}_1)^2} < 0$ for all $c < 1$, and $\frac{\partial P^*}{\partial c} = \frac{\hat{\theta}_1}{1+r_A\hat{\theta}_1} > 0$. Therefore, $P^*$ is decreasing in the updated expected value of fundamentals. Moreover, lowering the marginal cost of early liquidation—that is, imposing a smaller penalty to the lender—lowers the borrower's default probability. Finally, $\frac{\partial P^*}{\partial r_A} = \frac{\hat{\theta}_1^2(1-c)}{(1+r_A\hat{\theta}_1)^2} > 0$ for all $c < 1$. An increase in the riskless interest rate raises default risk, *ceteris paribus*.

Regarding the equilibrium liquidity risk measure, from (25) it follows that $\frac{\partial \lambda^*}{\partial \hat{\theta}_1} = \frac{\hat{\theta}_1(E+L)[\hat{\theta}_1(k-r_A)-2]+kL}{[(k-r_A)\hat{\theta}_1-1]^2 L}$. Although in general this is of ambiguous sign, for small values of $k$ it is likely to be negative regardless of the value of $\hat{\theta}_1$. An improved posterior belief about the state of fundamentals would thus induce a smaller proportion of investors to liquidate early, reflecting improved confidence in the project's chances of success. Second, $\frac{\partial \lambda^*}{\partial E} = \frac{\hat{\theta}_1^2}{[(k-r_A)\hat{\theta}_1-1]L} < 0$ for all $\hat{\theta}_1$: the greater the contribution of the internal endowment to the project, the fewer outside investors will opt to liquidate early. Third, it is easy to check that $\frac{\partial \lambda^*}{\partial k} = \frac{-\hat{\theta}_1^3(E+L)+\hat{\theta}_1 L(r_A\hat{\theta}_1+1)}{[(k-r_A)\hat{\theta}_1-1]^2 L}$ is likely to be positive unless $E$ is very large. Thus, liquidity risk is increasing in the disruption caused to the investment project by early withdrawals. However, if project finance relies more heavily on internal endowment (large $E$) then $\lambda^*$ can be decreasing in $k$. Internal funding then exerts a mitigating influence on liquidity risk.

Fourth, for small values of the riskless rate $r_A$ I obtain $\frac{\partial \lambda^*}{\partial L} > 0$, implying that more aggregate lending induces a larger percentage of investors to liquidate early, and *vice versa*. The pro-cyclicality of liquidity risk—positive

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correlation between $\lambda^*$ and the business cycle—also follows from $\frac{\partial \lambda^*}{\partial E} < 0$, to the extent that the internal endowment's relative share of the project's finance is likely to increase during periods of recession. Early liquidation is therefore pro-cyclical provided outside lending is also, suggesting that short-term capital flow reversals are less likely in slowdown periods.

Finally, dividing $\frac{\partial P^*}{\partial \hat{\theta}_1} < 0$ by $\frac{\partial \lambda^*}{\partial \hat{\theta}_1}$ implies $\frac{\partial P^*}{\partial \lambda^*} > (<) 0$ for small (large) values of $k$. Given $\frac{\partial \lambda^*}{\partial L} > 0$, the default probability is therefore pro-cyclical if the marginal disruption to the investment project caused by early liquidation is small, and counter-cyclical if $k$ is large. Procyclicality of aggregate default risk is exerting a stabilising influence on the business cycle. In contrast, negative correlation between macroeconomic growth and default risk (or average credit quality) amplifies business cycle fluctuations.

The tentative implication for financial regulation is the desirability of *small* values of $k$ in order to maintain pro-cyclical default risk. One such policy could involve the provision of guarantees to the project in the event of early liquidation by outside investors. This seems consistent with the result that lowering the charge to lenders for early liquidation reduces equilibrium default risk. If the latter is pro-cyclical, then regulators aiming to maintain a stable investment environment should impose fewer restrictions on short-term capital flows in times of expansion and more during slowdown.

More generally, the cyclicality of default risk raises the question whether, on aggregate, outside credit provision reduces the liquidity risk element of investment projects.[10] Recall from equations (16) that an open range of fundamentals $(\theta_{min}, \theta^{MAX})$ was obtained such that liquidity risk is non-zero: $\lambda^* \in (0, 1)$. Substituting the loan rate as a function of the fundamental realisation from equation (17) into (16) one can establish the fundamental values outside this range. On the one extreme, the *unstable* range $\theta \leq \theta_{min}$ captures the case of liquidity crises amounting to total liquidation of outside

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[10]For example, see Cooper (1999) and Obstfeld (1998).

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funding ($\lambda^* = 1$). On the other extreme lies the *stable* range of fundamentals $\theta \geq \theta^{MAX}$ such that there is no early liquidation ($\lambda^* = 0$). The stable range is given by:

$$\theta^2[E + L^*(\theta)] - \theta r_A L^*(\theta) - L^*(\theta) > 0 , \tag{26}$$

where the equilibrium amount of lending $L^*$ is found by substituting $r_L(\theta)$ from (17) into (14):

$$L^*(\theta) = -\frac{\theta \ln \underline{b}}{2[\theta(c + r_A) + 1]} \tag{27}$$

The stable range of fundamentals is a non-linear function of $\theta$, suggesting that the equilibrium liquidity risk measure may change discontinuously in response to small shocks to fundamentals. A numerical simulation study of the resulting restrictions is the next stage of this investigation.

## 5   Conclusion

In this paper I studied the strategic interaction between many risk-averse lenders and a single borrower in a two-period investment project with long-term assets and short-term liabilities. The equilibrium analysis focussed on assessing the default probability and liquidity risk for the cases of complete and incomplete information about the underlying fundamentals. The proportion of lenders liquidating early was characterised as a function of the credit spread, the lenders' penalty for early withdrawal and the disruption it causes the project, the contribution of the borrower's internal endowment, and the probability distributions of fundamentals and lenders' risk aversion. The cyclical properties of default and liquidity risk and their implications for international financial regulation were discussed and stable (unstable) ranges of fundamentals were derived such that liquidity risk is zero (one).

There are several directions in which the model could be extended. First, the stable and unstable ranges of fundamentals are sensitive to the underlying probability distribution functions; these were assumed to be uniform to simplify the exposition. Second, the lenders and borrower may plausibly have different beliefs about the unknown state of fundamentals. For example, faced with the same exogenous uncertainty, outside investors could be expected to be more pessimistic than the borrower. Third, fundamentals could deteriorate endogenously if enough outside funding is withdrawn early. Equation (1) could be generalised to consider time-varying fundamentals: $y_t = \theta_t(E + L_t)$. In a repeated game, persistent fundamentals may yield evolving equilibrium strategies capturing financial contagion dynamics, as defined by Allen and Gale (2000). Such extensions are the subject of future research.

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# Figure 1
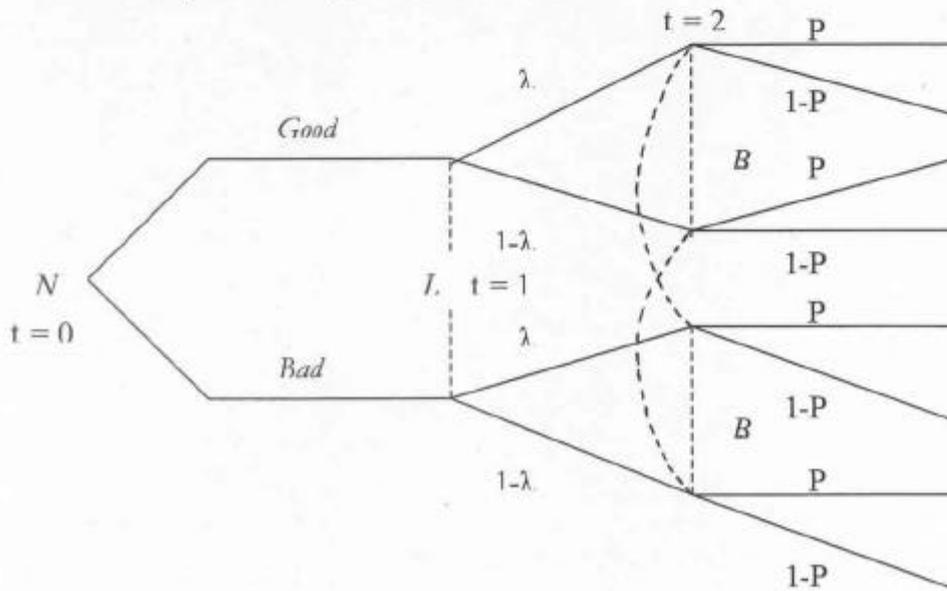## The extensive form games[*]

A. Complete and perfect information
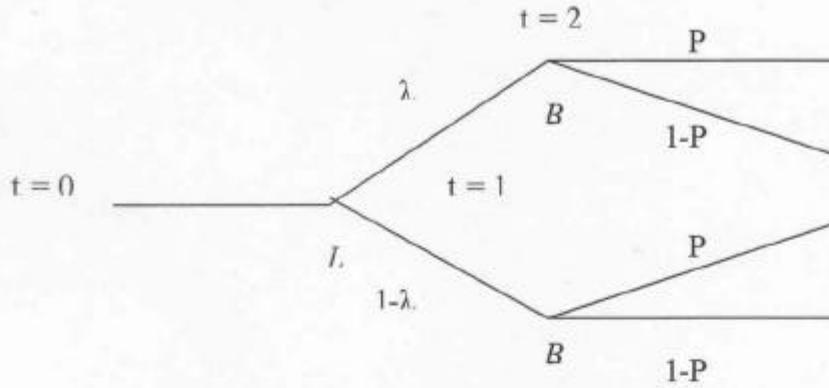


B. Complete and imperfect information
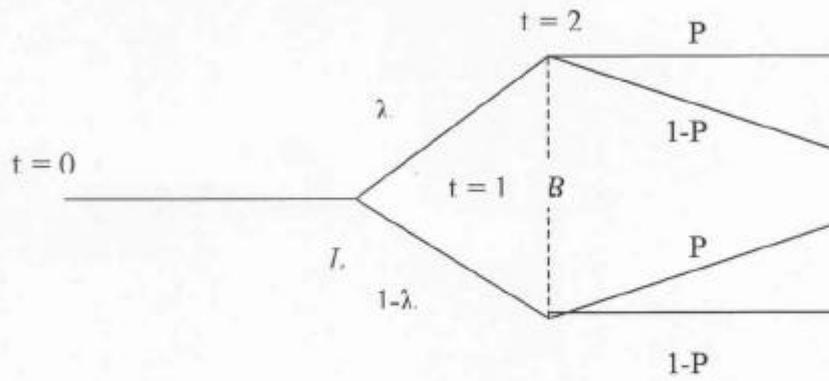


C. Incomplete and imperfect information



[*] The dotted lines link decision nodes that are not singletons. The payoffs at $t=2$ are given in Table 1.