MUTUALLY BENEFICIAL CONTINGENT CONTRACTS

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Abstract. We consider an incomplete market model with one traded risky asset and two Brownian motions. In this market, two small agents, each endowed with an initial capital and an illiquid position in a non-attainable claim and having preferences dictated by exponential utility functions, consider entering into a contingent contract that improves their positions in terms of expected utility. We find explicit expressions for a family of mutually beneficial contingent contracts and propose criteria for selecting a single "optimal" contract in this family.

1. Introduction

Consider a financial market model in a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ supporting two correlated Brownian motions $\mathbf{W}$ and $\tilde{\mathbf{W}}$, with $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ and $\mathcal{F} := \mathcal{F}_T$, and let $\tilde{\mathbb{F}} := \left(\tilde{\mathcal{F}}_t\right)_{t\in[0,T]}$ be the completion of the σ-field generated by $\tilde{\mathbf{W}}$ alone, with $\tilde{\mathcal{F}} := \tilde{\mathcal{F}}_T$. In this market we have only one risky traded asset, $\mathbf{S}$, which is driven by $\mathbf{W}$ and a locally riskless asset $\mathbf{B}$. It follows that the market is *incomplete*, since not every potential contingent claim can be replicated by trading in $\mathbf{S}$ and $\mathbf{B}$.

Consider first a market agent endowed with an initial capital x and an illiquid position in an asset that pays an $\tilde{\mathcal{F}}$-measurable amount $\mathbf{K}$ at time T, and having exponential utility function U. The problem faced by this agent is to maximize his expected utility; that is

$$\sup_{\pi\in\mathcal{A}} \mathbb{E}\left[U\left(\mathbf{X}_T^{\pi,x} + \mathbf{K}\right)\right],$$

where $\mathbf{X}_T^{\pi,x}$ is a self-financing portfolio following a trading strategy process π and $\mathcal{A}$ is the set of admissible portfolios.

Consider now two utility-maximizing agents, $\mathbf{A}$ and $\mathbf{B}$, endowed with initial capital x^A and x^B, having exponential utility functions U^A and U^B, and holding positions in illiquid, non-attainable assets with time-T payoffs represented by the $\tilde{\mathcal{F}}$-measurable random variables $\mathbf{K}^A$ and $\mathbf{K}^B$ respectively.

In this paper we are interested in identifying potential *contingent contracts* between these two agents, where agent $\mathbf{B}$ pays agent $\mathbf{A}$ at time T an amount given by the random variable $\mathbf{X}$, which can be positive or negative, so that the expected utilities of both agents increase.

In other words, we seek to characterize the potential contingent contracts $\mathbf{X}$ such that

$$\sup_{\pi\in\mathcal{A}} \mathbb{E}\left[U^A\left(\mathbf{X}_T^{\pi,x^A} + \mathbf{K}^A + \mathbf{X}\right)\right] > \sup_{\pi\in\mathcal{A}} \left[U^A\left(\mathbf{X}_T^{\pi,x^A} + \mathbf{K}^A\right)\right]$$

and

$$\sup_{\pi\in\mathcal{A}} \mathbb{E}\left[U^B\left(\mathbf{X}_T^{\pi,x^B} + \mathbf{K}^B - \mathbf{X}\right)\right] > \sup_{\pi\in\mathcal{A}} \mathbb{E}\left[U^B\left(\mathbf{X}_T^{\pi,x^B} + \mathbf{K}^B\right)\right].$$

The analysis of optimal financial contracts between utility maximizing economic agents starts with a series of papers by Borch in the early 60s (see Borch (1960a), Borch (1960b) and especially Borch (1962)), where he analyzes the problem of defining optimal reinsurance treaties. Although Borch's papers focus on reinsurance markets, the concepts are applicable to any market where agents seek to define an optimal distribution of risk. An important result, often referred as "Borch's theorem", is that a necessary and sufficient condition for a risk sharing agreement to be Pareto-optimal is that the

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ratio of the marginal utilities of any two contracting agents be constant. In general, there will be an infinite number of Pareto-optimal contracts, each defined by a particular choice for these constants; and Borch uses game theory to determine a unique agreement. In these papers agents are assumed to have perfect information, risks are modeled with random variables representing some final time payoffs, and there are no market assets in which the agents can trade.

After Borch's seminal papers, but still within the one-period framework, authors have aimed at both generalizing the original problem and providing further criteria for determining unique risk sharing contracts. Gerber (1978) assumed that some of the agents are not willing to contribute more than a certain fixed amount to the aggregate loss of the other companies. Buhlmann and Jewell (1979) added the actuarial concept of "long-run fairness" to each participant in the risk exchange, resulting in a unique Pareto-optimal risk transfer. In Buhlmann (1980), the concept of Pareto-optimal risk exchanges is related to the equilibrium pricing of such exchanges, and the equilibrium state-price density (the "economic premium principle") is derived explicitly for the exponential utility case. Buhlmann (1984) generalizes this economic premium principle to arbitrary risk averse utility functions. Wyler (1990) extends Buhlmann (1984) by a providing a bijectlve mapping between the set of Pareto optimal risk exchanges and the set of solutions of a system of differential equations. We note that these papers are concerned with identifying contracts that are Pareto-optimal, but not necessarily mutually beneficial. In Wyler (1990) this condition is dropped "to preserve the beauty of the main result".

In the early 2000s the topic of "risk transfer" is revisited in the context of derivative design rather than reinsurance agreements. Barrieu and El Karoui (2002a) looks at the problem of designing a derivative sold by a bank seeking to hedge its exposure to an illiquid position to an investor holding no initial risk. Both agents may invest their residual wealth on a financial market, but with the simplifying assumption that market trading decisions are independent from the illiquid risks. An explicit solution is provided for the optimal derivative structure in the exponential utility case, where the problem can be solved with standard convex optimization techniques. De Silvestro and Vargiolu (2002) show that similar results hold when the bank's optimization criterion is substituted with minimizing the expected shortfall, and with the variant that only the investor is allowed to trade in the financial market. Barrieu and El Karoui (2002b) applies the results from Barrieu and El Karoui (2002a) to the design of weather derivatives in particular.

In a series of papers (see Barrieu and El Karoui (2004a), Barrieu and El Karoui (2004b), Barrieu and El Karoui (2005) and Barrieu and El Karoui (2009)), the risk transfer problem from Barrieu and El Karoui (2002a) is recast in terms of the minimization of risk measures rather than maximization of utilities. In particular, agents assess their risk using convex risk measures. It is found that the inf-convolution of convex risk measures is the key transformation in solving the problem of designing optimal risk transfer contracts. For dynamic risk measures defined through their local specifications using BSDEs, their inf-convolution is equivalent to that of their associated drivers, making it also possible to characterize the optimal risk transfer. Explicit expressions for the optimal risk transfers are found for the entropic risk measures case, which is the equivalent to the exponential case in the utility framework. In Barrieu and Scandolo (2008), the authors expand the problem to include multi-period risks, where explicit results are shown for the exponential, time-additive utility functionals. Other recent papers on the subject include Burgert and Rüschendorf (2006), Zaphiropoulos and Zazanis (2006), Jouini et al. (2007), Dana and Scarsini (2007), Anthropelos and Zitkovic (2008), Burgert and Rüschendorf (2008), Filipovic and Kupper (2008), Zhou (2010) and Horst et al. (2010).

In this paper we take advantage of the explicit solutions to the portfolio optimization problem for the exponential utility case from Tehranchi (2004), for which we provide an alternative proof, to arrive at explicit expressions for the set of mutually beneficial contracts using standard convex optimization techniques. We identify the set of *equilibrium contracts*; that is, contracts such that once $\mathbf{X}$ has been added to the portfolios of agents $\mathbf{A}$ and $\mathbf{B}$, there are no further mutually beneficial contracts to be created between the agents, and we propose as a possible criterion to select a single equilibrium contract the *relative utility optimal contract*, which is defined as the equilibrium contract such that the relative increase in expected utilities for the two agents is the same.

This paper is structured as follows. In Section 2 we review the relevant results for the utility maximization problem for one agent from Tehranchi (2004) and provide an alternative proof of these results using convex duality. In Section 3 we develop the main problem considered in this paper of identifying optimal potential contracts between two agents. Finally in Section 4 we comment on related topics for future research.

2. Utility maximization problem for one agent

In this section we review the main results from Tehranchi (2004) for the problem of maximizing the expected utility of an agent with exponential utility function.

We start with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting the correlated Brownian motions $\mathbf{W}$ and $\tilde{\mathbf{W}}$, which have constant correlation ρ with $|\rho| < 0$. We denote by $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ the completion of the filtration generated by the pair $\left(\mathbf{W}, \tilde{\mathbf{W}}\right)$, with $\mathcal{F} := \mathcal{F}_T$; and $\tilde{\mathbb{F}} := \left(\tilde{\mathcal{F}}_t\right)_{t\in[0,T]}$ the completion of the filtration generated by $\tilde{\mathbf{W}}$ alone. The traded risky asset $\mathbf{S}$ follows the following dynamics:

$$d\mathbf{S}_t = S_t\left(\mu_t dt + \sigma_t d\mathbf{W}_t\right), \qquad \mathbf{S}_0 = s_0;$$

and the locally riskless asset $\mathbf{B}$ follows

$$dB_t = B_t r dt, \qquad B_0 = 1.$$

The Sharpe ratio process is defined as

$$\lambda_t := \frac{\mu_t - r}{\sigma_t}.$$

The model coefficients μ and σ are stochastic, such that the Sharpe ratio is $\tilde{\mathcal{F}}_t$-measurable for all $t \geq 0$, and $\sigma > 0$. The interest rate r is assumed to be constant.

The agent holds a self-financing portfolio with an amount of currency π invested in the risky asset and $\pi^{\mathbf{0}}$ invested in the locally riskless asset. The value of the portfolio is thus given by $\mathbf{X}_t = \pi_t + \pi_t^{\mathbf{0}}$, and by the self-financing condition we have that the portfolio evolves with dynamics

$$d\mathbf{X}_t^{\pi,x_0} = \pi_t\sigma_t\left(\lambda_t dt + d\mathbf{W}_t\right) + X_t^{\pi,x} r dt, \qquad \mathbf{X}_0 = x_0,$$

and we have

$$\mathbf{X}_t^{\pi,x_0} = e^{rt}\left(x_0 + \int_0^t e^{-ru}\pi_u\sigma_u\left(\lambda_u du + d\mathbf{W}_u\right)\right).$$

The agent's utility at time T is given by an exponential utility function

$$U(x) = -e^{-\gamma x},\ \gamma > 0.$$

Apart from the initial capital x, the agent is endowed with an illiquid position in a contract that pays a random amount $\mathbf{K}$ at time T, with $\mathbb{E}\left[\mathbf{K}^n\right] < \infty$ for all $n \in \mathbb{N}$.

The portfolio optimization problem is given by

$$\sup_{\pi\in\mathcal{A}} \mathbb{E}\left[U\left(\mathbf{X}_T^{\pi,x_0} + \mathbf{K}\right)\right],$$

where

$$\mathcal{A} = \left\{\pi : \pi \text{ is progressively measurable with } \mathbb{E}\left[\sup_{t\in[0,T]} e^{-\gamma' X_t^{\pi,x_0}}\right] < \infty \text{ for some } \gamma' > \gamma\right\}. \tag{2.1}$$

Proposition 1. *The maximum expected utility is given by*

$$\sup_{\pi\in\mathcal{A}} \mathbb{E}\left[U\left(\mathbf{X}_T^{\pi,x_0} + \mathbf{K}\right)\right] = -\exp\left\{-\gamma e^{rT}x_0\right\}\left(\mathbb{E}^{\tilde{M}}\left[\exp\left\{-\nu\left(\gamma\mathbf{K} + \frac{1}{2}\kappa\right)\right\}\right]\right)^{1/\nu}, \tag{2.2}$$

where $\kappa := \int_0^T \lambda_u^2 du$, $\nu := \left(1-\rho^2\right)$ *and* $\mathbb{E}^{\tilde{M}}[\cdot]$ *is the expectation with respect to the measure* $\tilde{\mathbb{P}}^M$ *given by* $\frac{d\tilde{\mathbb{P}}^M}{d\mathbb{P}} := \exp\left\{-\frac{\rho^2}{2}\int_0^T \lambda_s^2 ds - \rho\int_0^T \lambda_s d\tilde{\mathbf{W}}_s\right\}$.

For the proof we refer the reader to Proposition 3.3 in Tehranchi (2004), where we take $\mathbf{Y} = \exp\{-\gamma \mathbf{K}\}$. We now show how the same result can be obtained through convex duality techniques, but first we need some definitions.

Equivalent Local Martingale Measures. Let $\mathbf{W}^{\perp}$ be a Brownian motion orthogonal to $\mathbf{W}$ and $\tilde{\mathbf{W}}^{\perp}$ a Brownian motion orthogonal to $\tilde{\mathbf{W}}$, both $\mathbf{W}^{\perp}$ and $\tilde{\mathbf{W}}^{\perp}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$, so that we can write

$$\begin{bmatrix} \mathbf{W} \\ \mathbf{W}^{\perp} \end{bmatrix} = \begin{bmatrix} \rho & \sqrt{1-\rho^2} \\ -\sqrt{1-\rho^2} & \rho \end{bmatrix} \begin{bmatrix} \tilde{\mathbf{W}} \\ \tilde{\mathbf{W}}^{\perp} \end{bmatrix}. \tag{2.3}$$

Let $\mathcal{M}$ be the set of all equivalent local martingale measures (ELMMs) in $\mathcal{F}$. An arbitrary ELMM $\mathbb{Q} \in \mathcal{M}$ in our model is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \mathcal{E}\left(-\int_0^T \lambda_s d\mathbf{W}_s - \int_0^T \theta_s d\mathbf{W}_s^{\perp}\right),$$

where $\mathcal{E}$ is the Doléans exponential and θ is $\mathbb{F}$-adapted. The *minimal martingale measure* $\mathbb{P}^M$ is defined by taking $\theta \equiv 0$; that is,

$$\frac{d\mathbb{P}^M}{d\mathbb{P}} = \mathcal{E}\left(-\int_0^T \lambda_s d\mathbf{W}_s\right).$$

The projection of $\mathbb{P}^M$ onto $\tilde{\mathcal{F}}$ is denoted $\tilde{\mathbb{P}}^M$ and is given by

$$\frac{d\tilde{\mathbb{P}}^M}{d\mathbb{P}} = \mathcal{E}\left(-\int_0^T \rho\lambda_s d\tilde{\mathbf{W}}_s\right).$$

We will denote $\mathbb{E}^{\tilde{M}}[\cdot]$ the expectation with respect to $\tilde{\mathbb{P}}^M$. We can then decompose any $\mathbb{Q} \in \mathcal{M}$ as

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{d\tilde{\mathbb{P}}^M}{d\mathbb{P}} \frac{d\mathbb{Q}}{d\mathbb{P}^{\tilde{M}}},$$

where

$$\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}^M} = \mathcal{E}\left(-\int_0^T \left(-\sqrt{1-\rho^2}\right)\theta_s d\tilde{\mathbf{W}}_s - \int_0^T \left(\sqrt{1-\rho^2}\lambda_s + \rho\theta_s\right) d\tilde{\mathbf{W}}_s^{\perp}\right).$$

We will also need the minimal entropy martingale measure, $\mathbb{P}^E$, given by

$$\mathbb{P}^E := \arg\min_{\mathbb{Q}\in\mathcal{M}} \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right].$$

From Henderson et al. (2003), we have

$$\mathbb{E}\left[\frac{d\mathbb{P}^E}{d\mathbb{P}} \ln\left(\frac{d\mathbb{P}^E}{d\mathbb{P}}\right)\right] = -\frac{1}{1-\rho^2} \ln \mathbb{E}\left[\frac{d\tilde{\mathbb{P}}^M}{d\mathbb{P}} \exp\left(-\frac{1}{2}\left(1-\rho^2\right)\int_0^T \lambda_s^2 ds\right)\right]. \tag{2.4}$$

Portfolio optimization through convex duality. We start from the primal optimization problem

$$V(x_0, \mathbf{K}) = \sup_{\mathbf{X}} \mathbb{E}\left[U(\mathbf{X}, \mathbf{K})\right], \tag{2.5}$$

with

$$U(x,k) = -\exp\{-\gamma(x+k)\},$$

subject to $\mathbf{X}$ being a time-T claim in the space of random variables with finite moments that is attainable with initial capital x_0. If the market was complete, the constraint that $\mathbf{X}$ be attainable could be expressed as $\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} B_T^{-1}\mathbf{X}\right] \leq x_0$, where $\mathbb{Q}$ is the equivalent martingale measure (EMM). One could then solve the dual problem

$$V(x_0, \mathbf{K}) = \inf_{y>0} \mathbb{E}\left[\tilde{U}\left(y\frac{d\mathbb{Q}}{d\mathbb{P}} B_T^{-1}, \mathbf{K}\right)\right] + yx_0,$$

where

$$\begin{aligned}\tilde{U}(y,k) &= \sup_{x\in\mathbb{R}}\{U(x,k)-yx\}, \quad y>0\\ &= \frac{y}{\gamma}\left(\ln\left(\frac{y}{\gamma}\right)-1\right)+yk\end{aligned}$$

is the convex conjugate of U. This is the approach started in Karatzas et al. (1987). In our market model, however, the set of EMMs, $\mathcal{M}$, contains more than one element. As shown in Karatzas et al. (1991), an arbitrary $\mathbb{Q}\in\mathcal{M}$ can be interpreted as the unique EMM corresponding to the completion of the market by introducing a set of fictitious assets; and we look for the *least-favourable* market completion, corresponding to the case where the agent does not invest in the fictitious assets at all. This leads us the dual problem

$$\inf_{y>0,\mathbb{Q}\in\mathcal{M}}\mathbb{E}\left[\tilde{U}\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}B_T^{-1},\mathbf{K}\right)\right]+yx_0, \tag{2.6}$$

which we now proceed to solve. Direct calculation yields

$$\begin{aligned}\inf_{\mathbb{Q}\in\mathcal{M}}\mathbb{E}\left[\tilde{U}\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}B_T^{-1},\mathbf{K}\right)\right] &= yB_T^{-1}\inf_{\mathbb{Q}\in\mathcal{M}}\left\{\frac{1}{\gamma}\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right]+\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\mathbf{K}\right]\right\}\\ &+ \frac{yB_T^{-1}}{\gamma}\left(\ln\left(\frac{yB_T^{-1}}{\gamma}\right)-1\right).\end{aligned} \tag{2.7}$$

Here is where the assumption of $\tilde{\mathcal{F}}$-measurability of $\mathbf{K}$ and $\tilde{\mathbb{F}}$-adaptability of λ become necessary. We have

$$\begin{aligned}\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\mathbf{K}\right] &= \mathbb{E}\left[\frac{d\tilde{\mathbb{P}}^M}{d\mathbb{P}}\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}^M}\mathbf{K}\right]=\mathbb{E}\left[\mathbb{E}\left[\frac{d\tilde{\mathbb{P}}^M}{d\mathbb{P}}\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}^M}\mathbf{K}\mid\tilde{\mathcal{F}}\right]\right]\\ &= \mathbb{E}\left[\frac{d\tilde{\mathbb{P}}^M}{d\mathbb{P}}\mathbf{K}\mathbb{E}\left[\frac{d\mathbb{Q}}{d\tilde{\mathbb{P}}^M}\mid\tilde{\mathcal{F}}\right]\right]=\mathbb{E}^{\tilde{M}}[\mathbf{K}],\end{aligned}$$

which is independent from $\mathbb{Q}$, and so the optimization in (2.7) can be solved with (2.4). We have

$$\begin{aligned}\inf_{\mathbb{Q}\in\mathcal{M}}\mathbb{E}\left[\tilde{U}\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}B_T^{-1},\mathbf{K}\right)\right]+yx_0 &= y\left\{-B_T^{-1}\frac{1}{\gamma}\frac{1}{1-\rho^2}\ln\mathbb{E}^{\tilde{M}}\left[\exp\left(-\frac{1}{2}\left(1-\rho^2\right)\int_0^T\lambda_s^2ds\right)\right].\right.\\ &\left.+B_T^{-1}\mathbb{E}^{\tilde{M}}[\mathbf{K}]+\frac{B_T^{-1}}{\gamma}\left(\ln\left(\frac{yB_T^{-1}}{\gamma}\right)-1\right)+x_0\right\}.\end{aligned}$$

We now have a straightforward minimization over $y>0$, which yields (2.2).

3. Mutually beneficial contracts between two agents

Consider now two agents, **A** and **B**, endowed with initial capital x_0^A and x_0^B, having exponential utility functions with parameters γ^A and γ^B, and holding positions in illiquid, generally non-attainable assets with time-T payoffs represented by the $\tilde{\mathcal{F}}$-measurable random variables $\mathbf{K}^A$ and $\mathbf{K}^B$ respectively, where $\mathbb{E}\left[\exp\left\{-\gamma^A\mathbf{K}^An\right\}\right]<\infty$ and $\mathbb{E}\left[\exp\left\{-\gamma^B\mathbf{K}^Bn\right\}\right]<\infty$ for all $n\in\mathbb{N}$. The admissible strategies for the agents, $\mathcal{A}^A$ and $\mathcal{A}^B$, are defined similar to (2.1) but with the corresponding initial capital and utility parameter.

We are interested in identifying potential contingent contracts between these two agents, where agent **B** pays agent **A** at time T an amount given by the random variable $\mathbf{X}$, which can be positive or negative, so that the expected utilities of both agents increase. The following definitions make these notions more precise.

Definition 2. A contract with payoff $\mathbf{X}$ is said to be *admissible* if $\mathbf{X}$ is $\tilde{\mathcal{F}}$-measurable and if $\mathbb{E}\left[\exp\left\{-\gamma^A\left(\mathbf{K}^A-\mathbf{X}\right)n\right\}\right]<\infty$ and $\mathbb{E}\left[\exp\left\{-\gamma^B\left(\mathbf{K}^B-\mathbf{X}\right)n\right\}\right]<\infty$ for all $n\in\mathbb{N}$. We denote $\mathcal{X}$ the set of all admissible contracts.

Definition 3. An admissible contract $\mathbf{X}$ is said to be *mutually beneficial* if

$$\sup_{\pi\in\mathcal{A}^A}\mathbb{E}\left[U^A\left(\mathbf{X}_T^{\pi,x_0^A}+\mathbf{K}^A+\mathbf{X}\right)\right]\geq\sup_{\pi\in\mathcal{A}^A}\mathbb{E}\left[U^A\left(\mathbf{X}_T^{\pi,x_0^A}+\mathbf{K}^A\right)\right]$$

and

$$\sup_{\pi\in\mathcal{A}^B}\mathbb{E}\left[U^B\left(\mathbf{X}_T^{\pi,x_0^B}+\mathbf{K}^B-\mathbf{X}\right)\right]\geq\sup_{\pi\in\mathcal{A}^B}\mathbb{E}\left[U^B\left(\mathbf{X}_T^{\pi,x_0^B}+\mathbf{K}^B\right)\right].$$

When both inequalities are strict, we say that the contract is *strictly mutually beneficial*. We denote $\mathcal{X}_b$ the set of all strictly mutually beneficial contracts.

Definition 4. An admissible contract $\mathbf{X}$ is said to be an *equilibrium* contract if there is no strictly mutually beneficial contract of the form $\mathbf{X}':=\mathbf{X}+\mathbf{Y}$ for any $\mathbf{Y}\in\mathcal{X}$. In other words, an admissible contract $\mathbf{X}$ is said to be an equilibrium contract if, once $\mathbf{X}$ is added to the portfolios of agents $\mathbf{A}$ and $\mathbf{B}$, there are no further strictly mutually beneficial contracts. We denote $\mathcal{X}_e$ the set of all admissible equilibrium contracts and $\mathcal{X}_{b,e}$ the set of all admissible equilibrium contracts that are also strictly mutually beneficial.

Definition 5. We define three specific contracts. We denote $\mathbf{X}_{ind}^A$ the mutually beneficial equilibrium contract such that agent $\mathbf{A}$ is indifferent. We will see that this is the best possible equilibrium contract agent $\mathbf{B}$ can hope for that does not decrease $\mathbf{A}$'s expected utility. Similarly, we denote $\mathbf{X}_{ind}^B$ the mutually beneficial equilibrium contract such that agent $\mathbf{A}$ is indifferent. Finally, we denote $\mathbf{X}^*$the mutually beneficial equilibrium contract such that the relative increase in expected utility for both agents is the same, and we call it the *relative utility optimal contract*.

We now attack the problem of characterizing the sets $\mathcal{X}_e$ and $\mathcal{X}_{b,e}$, and the contracts $\mathbf{X}_{ind}^A$, $\mathbf{X}_{ind}^B$ and $\mathbf{X}^*$. Consider the optimization problem

$$C^A\left(c^B\right):=\left\{\inf_{\mathbf{X}\in\mathcal{X}}\frac{\sup_{\pi\in\mathcal{A}^A}\mathbb{E}\left[U^A\left(\mathbf{X}_T^{\pi,x_0^A}+\mathbf{K}^A+\mathbf{X}\right)\right]}{\sup_{\pi\in\mathcal{A}^A}\mathbb{E}\left[U^A\left(\mathbf{X}_T^{\pi,x_0^A}+\mathbf{K}^A\right)\right]}\right.$$

$$\left.\text{subject to }\frac{\sup_{\pi\in\mathcal{A}^B}\mathbb{E}\left[\left\{U^B\left(\mathbf{X}_T^{\pi,x_0^B}+\mathbf{K}^B-\mathbf{X}\right)\right\}\right]}{\sup_{\pi\in\mathcal{A}^B}\mathbb{E}\left[U^B\left(\mathbf{X}_T^{\pi,x_0^B}+\mathbf{K}^B\right)\right]}\leq c^B\right\}. \tag{3.1}$$

The constant c^B specifies the optimization constraint in terms of the ratio between the expected utilities of $\mathbf{B}$ with and without the contract $\mathbf{X}$. Since the expected utilities are always negative, $\mathbf{B}$ wants c^B to be as small as possible. c^B can take values in $(0,\infty)$. The contracts satisfying the constraint will increase $\mathbf{B}$'s expected utility when $c^B\in(0,1)$ and decrease it when $c^B\in(1,\infty)$. When $c^B=1$, $\mathbf{B}$ is indifferent with respect to these feasible contracts.

For a given c^B, $C^A\left(c^B\right)$ provides the ratio representing the largest increase in expected utility for $\mathbf{A}$. Similarly to $\mathbf{B}$'s case, $\mathbf{A}$ wants C^A to be as small as possible, with $C^A=1$ representing the break-even ratio.

We proceed to solve (3.1). To keep notation short, we define the functional

$$\Lambda(\cdot):=\left(\mathbb{E}^{\mathbb{Q}}\left[\exp\left\{-\nu\left(\cdot+\frac{1}{2}\kappa\right)\right\}\right]\right)^{1/\nu}, \tag{3.2}$$

and write (2.2) as

$$\sup_{\pi\in\mathcal{A}}\mathbb{E}\left[U\left(\mathbf{X}_T^{\pi,x_0}+\mathbf{K}\right)\right]=-\exp\left\{-\gamma e^{rT}x_0\right\}\Lambda\left(\gamma\mathbf{K}\right). \tag{3.3}$$

From (3.3) we have

$$C^A\left(c^B\right)=\frac{1}{\Lambda\left(\gamma^A\mathbf{K}^A\right)}\left\{\inf_{\mathbf{X}\in\mathcal{X}}\Lambda\left(\gamma^A\left(\mathbf{K}^A+\mathbf{X}\right)\right)\text{ subject to }\frac{\Lambda\left(\gamma^B\left(\mathbf{K}^B-\mathbf{X}\right)\right)}{\Lambda\left(\gamma^B\mathbf{K}^B\right)}\leq c^B\right\}.$$

Define the Lagrangean functional

$$\begin{aligned} L(\mathbf{X}, y) : &= \mathbb{E}\left[\exp\left\{-\nu\left(\gamma^A\left(\mathbf{K}^A + \mathbf{X}\right) + \frac{1}{2}\kappa\right)\right\},\right. \\ &\left. +y\left(\exp\left\{-\nu\left(\gamma^B\mathbf{K}^B + \frac{1}{2}\kappa\right)\right\}\left(\exp\left\{\gamma^B\nu\mathbf{X}\right\} - c^\nu\right)\right)\right] \end{aligned}$$

and define the function

$$\begin{aligned} l(x, y, \omega) : &= \exp\left\{-\nu\left(\gamma^A\left(\mathbf{K}^A(\omega) + x\right) + \frac{1}{2}\kappa\right)\right\} \\ &+y\left(\exp\left\{-\nu\left(\gamma^B\mathbf{K}^B(\omega) + \frac{1}{2}\kappa\right)\right\}\left(\exp\left\{\gamma^B\nu x\right\} - c^\nu\right)\right). \end{aligned}$$

We have

$$L(\mathbf{X}, \lambda) = \int_\Omega l(\mathbf{X}(\omega), \lambda, \omega)\, d\mathbb{P}(\omega),$$

and

$$\inf_{\mathbf{X}\in\mathcal{X}} L(\mathbf{X}, \lambda) = \int_\Omega \inf_{x\in\mathbb{R}} l(x, \lambda, \omega)\, d\mathbb{P}(\omega).$$

To solve $\inf_{x\in\mathbb{R}} l(x, \lambda, \omega)$ we calculate

$$\begin{aligned} \frac{\partial l}{\partial x} &= -\nu\gamma^A \exp\left\{-\nu\left(\gamma^A\left(\mathbf{K}^A(\omega) + x\right) + \frac{1}{2}\kappa\right)\right\}, \\ &+y\gamma^B\nu\exp\left\{-\nu\left(\gamma^B\mathbf{K}^B(\omega) + \frac{1}{2}\kappa\right)\right\}\exp\left\{\gamma^B\nu x\right\}, \end{aligned}$$

and equating to zero and solving for x we can define the random variable

$$\hat{\mathbf{X}} := \frac{\left(\gamma^B\mathbf{K}^B - \gamma^A\mathbf{K}^A\right)}{\left(\gamma^A + \gamma^B\right)} + \ln\left(y\frac{\gamma^B}{\gamma^A}\right)^{-\frac{1}{\nu\left(\gamma^A+\gamma^B\right)}}. \tag{3.4}$$

We now find the value y that makes $\hat{\mathbf{X}}$ feasible for the optimization problem by equating

$$\frac{\Lambda\left(\gamma^B\left(\mathbf{K}^B - \hat{\mathbf{X}}\right)\right)}{\Lambda\left(\gamma^B\mathbf{K}^B\right)} = c^B,$$

and solving for y. We obtain

$$\hat{y} := \frac{\gamma^A}{\gamma^B}\left(c^B\frac{\Lambda\left(\gamma^B\mathbf{K}^B\right)}{\Lambda\left(\frac{\gamma^B}{(\gamma^A+\gamma^B)}\gamma^A\mathbf{K}^A + \frac{\gamma^A}{(\gamma^A+\gamma^B)}\gamma^B\mathbf{K}^B\right)}\right)^{-\frac{\nu\left(\gamma^A+\gamma^B\right)}{\gamma^B}}. \tag{3.5}$$

Replacing (3.5) in (3.4), we have that the $\mathbf{A}$'s optimal contract for a given c^B is given by

$$\hat{\mathbf{X}}\left(c^B\right) := \frac{\left(\gamma^B\mathbf{K}^B - \gamma^A\mathbf{K}^A\right)}{\left(\gamma^A + \gamma^B\right)} + \ln\left(\left(c^B\right)^{\frac{1}{\gamma^B}}\Pi^B\right), \tag{3.6}$$

where

$$\Pi^B := \left(\frac{\Lambda\left(\gamma^B\mathbf{K}^B\right)}{\Lambda\left(\frac{\gamma^B}{(\gamma^A+\gamma^B)}\gamma^A\mathbf{K}^A + \frac{\gamma^A}{(\gamma^A+\gamma^B)}\gamma^B\mathbf{K}^B\right)}\right)^{\frac{1}{\gamma^B}}.$$

We finally obtain

$$C^A\left(c^B\right) = \frac{\Lambda\left(\gamma^A\left(\mathbf{K}^A+\hat{\mathbf{X}}\left(c^B\right)\right)\right)}{\Lambda\left(\gamma^A\mathbf{K}^A\right)}$$

$$= \frac{1}{\left(c^B\right)^{\frac{\gamma^A}{\gamma^B}}}\left(\frac{\Pi^A}{\Pi^B}\right)^{\gamma^A}, \tag{3.7}$$

where

$$\Pi^A := \left(\frac{\Lambda\left(\gamma^A\mathbf{K}^A\right)}{\Lambda\left(\frac{\gamma^B}{\left(\gamma^A+\gamma^B\right)}\gamma^A\mathbf{K}^A+\frac{\gamma^A}{\left(\gamma^A+\gamma^B\right)}\gamma^B\mathbf{K}^B\right)}\right)^{-\frac{1}{\gamma^A}}.$$

Let us gain intuition on these results. From (3.7), we see that C^A is monotonically decreasing on c^B, representing the trade-off between the preferences of the two agents. For $\hat{\mathbf{X}}\left(c^B\right)$ to be strictly mutually beneficial we need both $c^B \in (0,1)$ and $C^A\left(c^B\right) \in (0,1)$. Noting that $C^A(1) \le 1$ (show), and defining

$$c^B_{min} := c^B \text{ such that } C^A\left(c^B\right) = 1,$$

so that equating (3.7) to 1 and solving for c^B we obtain

$$c^B_{min} = \frac{\Lambda\left(\frac{\gamma^B}{\left(\gamma^A+\gamma^B\right)}\gamma^A\mathbf{K}^A+\frac{\gamma^A}{\left(\gamma^A+\gamma^B\right)}\gamma^B\mathbf{K}^B\right)^{\frac{\gamma^A+\gamma^B}{\gamma^A}}}{\Lambda\left(\gamma^A\mathbf{K}^A\right)^{\frac{\gamma^B}{\gamma^A}}\Lambda\left(\gamma^B\mathbf{K}^B\right)},$$

we see that for $\hat{\mathbf{X}}\left(c^B\right)$ to be strictly mutually beneficial we need $c^B \in \left(c^B_{min}, 1\right)$. We also have

$$\mathbf{X}^B_{ind} = \hat{\mathbf{X}}(1),$$
$$\mathbf{X}^A_{ind} = \hat{\mathbf{X}}\left(c^B_{min}\right).$$

Finally, by equating the right hand side of (3.7) with c^B we obtain

$$\mathbf{X}^* = \hat{\mathbf{X}}\left(c^*\right),$$

where

$$c^* := \left\{c^B \text{ such that } C^A\left(c^B\right) = c^B\right\} = \frac{\Lambda\left(\frac{\gamma^B}{\left(\gamma^A+\gamma^B\right)}\gamma^A\mathbf{K}^A+\frac{\gamma^A}{\left(\gamma^A+\gamma^B\right)}\gamma^B\mathbf{K}^B\right)}{\Lambda\left(\gamma^A\mathbf{K}^A\right)^{\frac{\gamma^B}{\gamma^A+\gamma^B}}\Lambda\left(\gamma^B\mathbf{K}^B\right)^{\frac{\gamma^A}{\gamma^A+\gamma^B}}}. \tag{3.8}$$

We illustrate these results in Figure 3.1.

Remark 6. We have approached the problem in (3.1) by optimizing **A**'s expected utilities given **B**'s constraint. This is arbitrary, as we could have solved the symmetrical problem of optimizing **B**'s expected utilities given **A**'s constraint, obtaining C^B as a function of c^A (which is the inverse function of $C^A\left(c^B\right)$. We now turn to a somewhat more satisfactory parametrization to formalize the main results.

A convenient parametrization. We note that the argument of the logarithm in (3.6) is a continuous, strictly increasing function of c^B. We are interested in the interval between $c^B = c^B_{min}$ and $c^B = 1$, where the argument of the logarithm takes values between Π^A and Π^B respectively. This justifies the following parametrization:

$$\alpha\Pi^B + (1-\alpha)\Pi^A = \left(c^B\right)^{\frac{1}{\gamma^B}}\Pi^B,$$

so that

$$c^B = \left(\frac{\alpha\Pi^B+(1-\alpha)\Pi^A}{\Pi^B}\right)^{\gamma^B}, \tag{3.9}$$

and replacing the right hand side of (3.9) in (3.7) we obtain

$$C^A = \left(\frac{\Pi^A}{\alpha \Pi^B + (1-\alpha)\Pi^A} \right)^{\gamma^A}. \tag{3.10}$$

Also,

$$\alpha = \frac{\left(c^B\right)^{\frac{1}{\gamma^B}} \Pi^B - \Pi^A}{\Pi^B - \Pi^A}. \tag{3.11}$$

Theorem 7. *The set $\mathcal{X}_{b,e}$ of admissible, equilibrium and strictly mutually beneficial contracts is given by all random variables $\hat{\mathbf{X}}$ of the form*

$$\hat{\mathbf{X}}(\alpha) = \frac{\gamma^B \mathbf{K}^B - \gamma^A \mathbf{K}^A}{\gamma^A + \gamma^B} + \ln\left[\alpha \Pi^B + (1-\alpha)\Pi^A\right], \tag{3.12}$$

where

$$\alpha \in (0,1).$$

The indifference contracts *are given by*

$$\mathbf{X}^B_{ind} = \hat{\mathbf{X}}(0),$$
$$\mathbf{X}^A_{ind} = \hat{\mathbf{X}}(1).$$

The relative utility optimal contract *is given by*

$$\mathbf{X}^* = \hat{\mathbf{X}}(\alpha^*),$$

where

$$\alpha^* = \frac{\left(\frac{\Lambda\left(\gamma^B \mathbf{K}^B\right)}{\Lambda\left(\gamma^A \mathbf{K}^A\right)} \right)^{\frac{1}{\gamma^A + \gamma^B}} - \Pi^A}{\Pi^B - \Pi^A}. \tag{3.13}$$

In particular, we have

$$\mathbf{X}^* = \frac{\gamma^B \mathbf{K}^B - \gamma^A \mathbf{K}^A}{\gamma^A + \gamma^B} + \frac{1}{\gamma^A + \gamma^B} \ln\left[\frac{\Lambda\left(\gamma^B \mathbf{K}^B\right)}{\Lambda\left(\gamma^A \mathbf{K}^A\right)} \right], \tag{3.14}$$

and the corresponding ratio is given by

$$c^* = \frac{\Lambda\left(\frac{\gamma^B}{(\gamma^A + \gamma^B)} \gamma^A \mathbf{K}^A + \frac{\gamma^A}{(\gamma^A + \gamma^B)} \gamma^B \mathbf{K}^B \right)}{\Lambda\left(\gamma^B \mathbf{K}^B\right)^{\frac{\gamma^A}{\gamma^A + \gamma^B}} \Lambda\left(\gamma^A \mathbf{K}^A\right)^{\frac{\gamma^B}{\gamma^A + \gamma^B}}}. \tag{3.15}$$

For a given α, the expected utility ratios corresponding to contract $\hat{\mathbf{X}}(\alpha)$ are given by (3.9) and (3.10).

Proof. The only things left to prove are (3.13) and (3.14). (3.13) follows from replacing (3.8) in (3.11). Then (3.14) follows from replacing (3.13) in (3.12). □

Figure 3.2 illustrates these concepts.

Example 8. As a very simple example, let $\mathbf{U}$ be the value of a non-tradeable factor with dynamics

$$d\mathbf{U}_t = U_t \left(m_t dt + v_t d\tilde{\mathbf{W}}_t \right), \qquad \mathbf{U}_0 = u_0,$$

and let $\mathbf{K}^A = (K - \mathbf{U}_T)^+$, $\mathbf{K}^B = (\mathbf{U}_T - K)^+$, with $K \in \mathbb{R}_+$, and $\gamma^A = \gamma^B = \gamma$. Then

$$\mathbf{X}(\alpha) = \frac{(\mathbf{U}_T - K)}{2} + \ln\left[\alpha \Pi^B + (1-\alpha)\Pi^A\right]. \tag{3.16}$$

So if one agent is exposed to an European call option and the other agent to an European put option on $\mathbf{U}_T$, they should enter into a forward contract.

FIGURE 3.1. C^A as a function of c^B

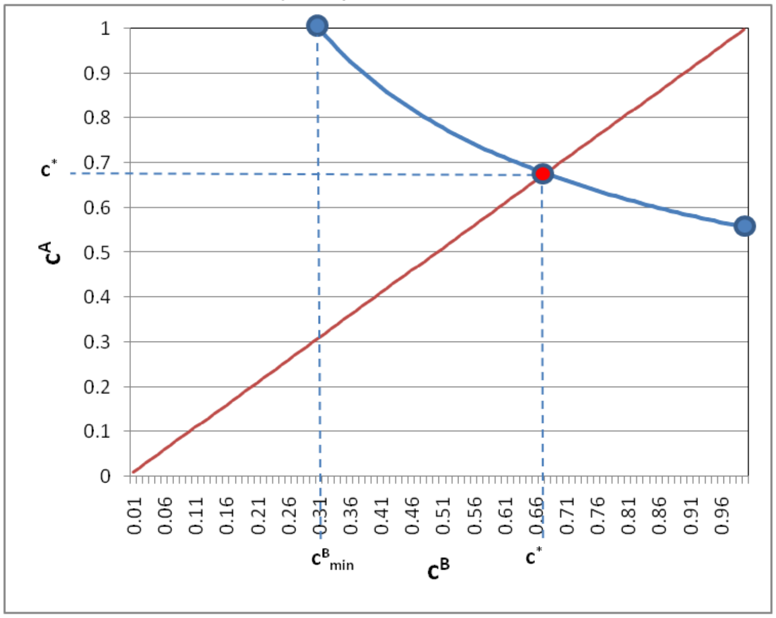
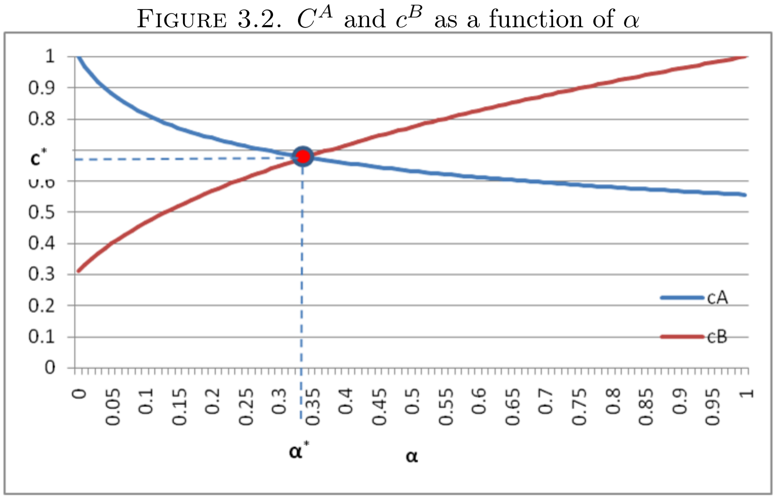


FIGURE 3.2. C^A and c^B as a function of α



4. Conclusions and ideas for future research

It is interesting to see that in an arbitrage-free model, the presence of economic agents with different risk positions gives rise to opportunities that are for these agents equivalent to arbitrage. To see this, we calculate the *certainty equivalent* of the mutually beneficial contracts:

$$\begin{aligned} h^A(\alpha) := \quad h \text{ such that } \quad & \sup_{\pi\in\mathcal{A}^A} \mathbb{E}\left[U^A\left(\mathbf{X}_T^{\pi,x_0^A} + \mathbf{K}^A + \hat{\mathbf{X}}(\alpha)\right)\right] \\ & = \sup_{\pi\in\mathcal{A}^A} \mathbb{E}\left[U^A\left(\mathbf{X}_T^{\pi,x_0^A} + \mathbf{K}^A + h\right)\right], \end{aligned}$$

$$h^B(\alpha) := \quad h \text{ such that} \quad \sup_{\pi\in\mathcal{A}^B} \mathbb{E}\left[U^B\left(\mathbf{X}_T^{\pi,x_0^B} + \mathbf{K}^B - \hat{\mathbf{X}}(\alpha)\right)\right] = \sup_{\pi\in\mathcal{A}^B} \mathbb{E}\left[U^B\left(\mathbf{X}_T^{\pi,x_0^B} + \mathbf{K}^B + h\right)\right].$$

Straightforward calculations yield

$$h^A(\alpha) = -\frac{1}{\gamma^A}\ln\left[\frac{\Lambda\left(\gamma^A\gamma^B\left(\frac{\mathbf{K}^A+\mathbf{K}^B}{\gamma^A+\gamma^B}\right) + \gamma^A\ln\left[\alpha\Pi^B + (1-\alpha)\Pi^A\right]\right)}{\Lambda\left(\gamma^A\mathbf{K}^A\right)}\right],$$

$$h^B(\alpha) = -\frac{1}{\gamma^B}\ln\left[\frac{\Lambda\left(\gamma^A\gamma^B\left(\frac{\mathbf{K}^A+\mathbf{K}^B}{\gamma^A+\gamma^B}\right) - \gamma^B\ln\left[\alpha\Pi^B + (1-\alpha)\Pi^A\right]\right)}{\Lambda\left(\gamma^B\mathbf{K}^B\right)}\right],$$

both of which are positive for $\alpha \in (0,1)$. This means that agent **A** (resp. **B**) is indifferent between entering into the contract $\hat{\mathbf{X}}(\alpha)$ (resp. $-\hat{\mathbf{X}}(\alpha)$) and receiving a risk-free positive cash amount of $h^A(\alpha)$ (resp. $h^B(\alpha)$).

If instead of allowing the agents to trade with themselves we assume that they do not know of each other's existence, and we introduce a financial intermediary that knows about these agents, the financial intermediary could sell $\hat{\mathbf{X}}(\alpha)$ to **A** for some $p^A < h^A(1)$ and sell $-\hat{\mathbf{X}}(\alpha)$ to **B** for some $p^B < h^B(0)$, such that $p^A + p^B > 0$, cashing in a risk-free cash amount $p = p^A + p^B \in \left(0, h^A(1) + h^B(0)\right)$. This justifies within the model the role of financial intermediation.

In this model we assumed the agents to be "small", in the sense that the prices of the market-traded assets are given exogenously and the behavior of the agents do not affect them. An interesting approach for future research is to investigate the case where the agents *are* the market, and thus determine the asset dynamics through market-clearing conditions. In this case, the market price of risk process λ will be determined endogenously; and the introduction of mutually beneficial contracts would affect it. Following this line of analysis, equivalent to introducing a mutually beneficial claim would be the introduction of a market-traded asset, purely "financial" in the sense that it does not derive its value from existing streams of cash flows, that effectively completes the market.

The present setting can be modified and/or generalized in multiple directions: the number of agents can be generalized to some $n \geq 2$, the type of contracts can be streams of cash flows rather than time-T payoffs, other objective functionals can be considered like the maximization of the instantaneous utility of consumption or dividends, the sources of market incompleteness could include transaction costs, trading restrictions, etc., there could be asymmetrical information such that not all agents can trade with each other, and agents could be allowed to hold different beliefs. We could also consider different bargaining games to determine specific contracts (e.g., to determine α). We plan to purse these ideas in future works.

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