Discussion of “Uses of random field theory for analyzing financial risk” by Erik Vanmarcke

Mike Tehranchi

Statistical Laboratory
University of Cambridge

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Quantum mass ratio.

The quantum mass ratio is a random variable with pdf

$$f(w) = \frac{aw^2}{e^{bw} - 1}, \quad w \geq 0$$

where the constants $a, b > 0$ are such that

$$\int_0^\infty f(w)dw = 1 \text{ and } \int_0^\infty \log(w)f(w)dw = 0$$

We can look at a more general class of pdf's

$$f(w) = \frac{aw^{s-1}}{e^{bw} - 1}, \quad w \geq 0$$

for $s > 1$.
(The quantum mass ratio corresponds to $s = 3$)

Proposition

If

$$a = \frac{b^s}{\Gamma(s)\zeta(s)} \quad \textit{and} \quad \log b = \frac{\Gamma'(s)}{\Gamma(s)} + \frac{\zeta'(s)}{\zeta(s)}$$

then

$$\int_0^\infty f(w)dw = 1 \quad \textit{and} \quad \int_0^\infty \log(w) f(w) dw = 0$$

where

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx \quad \textit{and} \quad \zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}.$$

The proposition is a consequence of the identity

$$\int_0^\infty \frac{x^{s-1}dx}{e^x - 1} = \int_0^\infty \sum_{n=1}^\infty x^{s-1} e^{-nx} dx$$

$$= \int_0^\infty \sum_{n=1}^\infty n^{-s} y^{s-1} e^{-y} dy$$

$$= \Gamma(s)\zeta(s)$$

In particular, if the random variable W has pdf f then the moments can be calculated

$$\mathbb{E}\ W^t = \frac{\Gamma(t+s)\zeta(t+s)}{b^t\Gamma(s)\zeta(s)}.$$

and even

$$\mathbb{E}\ (\log W)^n = \frac{1}{\Gamma(s)\zeta(s)} \sum_{k=0}^{n} \binom{n}{k} (-\log b)^{n-k} \frac{d^k}{ds^k}[\Gamma(s)\zeta(s)]$$

Proposition

Suppose the random variable W_s has the pdf with parameter s. Then

$$\sqrt{s}\log W_s \to N(0,1) \quad \text{as } s \to \infty.$$

Stationary stochastic processes.

Let $X = \{X(t) : t \in \mathbb{R}\}$ be a stochastic process, such that

$$\mathbb{E}[X(t)] = 0$$

and

$$\mathbb{E}[X(t)^2] = \sigma^2 < \infty$$

for all t.

Suppose that X is stationary in the sense that there is function ρ such that

$$\mathbb{E}[X(s)X(t)] = \sigma^2 \rho(t-s)$$

for all s, t.
Suppose ρ is integrable, and let

$$\theta = \int_{-\infty}^{\infty} \rho(u) du.$$

Define the locally averaged process

$$X_D(t) = \frac{1}{2D}\int_{-D}^{D} X(u+t)du$$

Then

$$\begin{aligned}\gamma(D) &= \frac{\mathbb{E}[X_D(t)^2]}{\mathbb{E}[X(t)^2]} \\ &= \frac{1}{D^2}\int_{-D}^{D}(D-|u|)\rho(u)du\end{aligned}$$

Proposition

$$\lim_{D \to \infty} D \, \gamma(D) = \theta.$$

Proof.

l'Hôpital's rule. □

Theorem

Suppose ρ is continuous. Then there exists a probability measure μ such that

$$\rho(t) = \int e^{itx} \mu(dx).$$

Proof.

Bochner's theorem characterises the Fourier transform of a finite measure. □

Theorem

There exists a pdf g such that

$$\mu(dx) = g(x)dx,$$

given by the formula

$$g(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\rho(t)dt.$$

In particular,

$$\theta = 2\pi g(0).$$

Proof.

This is the Fourier inversion formula. □

An application: random fields in finance

Let $P(t, T)$ denote the price at time t of a zero-coupon bond worth one unit of cash at time T.

$$f(t, T) = -\frac{\partial}{\partial T} \log P(t, T)$$

denotes the instantaneous forward rate.

The forward rate surface can be modelled as a Gaussian random field (Kennedy 1994):

$$\mathbb{E}[f(t,T)] = \mu(t,T)$$

$$\mathrm{Cov}[f(s,S), f(t,T)] = C(s,t;S,T)$$

Proposition

Suppose that covariance has the form

$$C(s, S; t, T) = c_{s \wedge t}(S, T).$$

Then, for each fixed $T > 0$, *the increments of* $(f(t, T))_{t \in [0,T]}$ *are independent.*

Theorem (Heath–Jarrow–Morton 1992, Kennedy 1994)

If

$$\mu(t, T) = f(0, T) + \int_0^T c_{t\wedge s}(s, T) ds.$$

then there is no arbitrage in the bond market with prices

$$P(t, T) = \exp\left(- \int_t^T f(t, u)\, du \right)$$

Interesting case Let

$$r_t(x) = f(t, t + x).$$

Suppose there is a space of functions F such that $r_t(\cdot, \omega) \in F$ for all (t, ω).

One can regard $(r_t)_{t \geq 0}$ as Gaussian Markov process valued in F.

Infinite-dimensional Ornstein–Uhlenbeck process

$$dr_t = \left(\frac{\partial r_t}{\partial x} + \mu \right) dt + \sigma dW_t$$

Sufficient conditions for ergodicity found in Vargiolou 1999, Tehranchi 2005.